Market Segmentation and Software Security: Pricing Patching Rights

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Abstract

The patching approach to security in the software industry has been less effective than desired. One critical issue with the status quo is that the endowment of "patching rights" (the ability for a consumer to choose whether security updates are applied) lacks the incentive structure to induce better security-related decisions. In this paper, we establish how producers can differentiate their products based on the provision of patching rights and how the optimal pricing of these rights can segment the market in a manner that leads to both greater security and greater profitability. We characterize the price for these rights, the discount provided to those who relinquish rights and have their systems automatically updated, and the consumption and protection strategies taken by users in equilibrium as they strategically interact due to the security externality associated with product vulnerabilities. We quantify the effectiveness of priced patching rights, its impact on welfare, and the ability of taxes to achieve an analogous effect in the open-source domain. In this domain, we demonstrate why large populations of unpatched users remain even when automatic updating is available, and then characterize how taxes on patching rights should optimally be structured.

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1 Introduction

Security attacks on unpatched software continue to be a major problem. The Department of Homeland Security's United States Computer Emergency Readiness Team indicates that systems running unpatched versions of software from providers such as Microsoft, Adobe, and OpenSSL are consistently attacked (US-CERT 2015). The failure of software users to patch in a timely manner enables attackers with malicious intentions to access their systems and oftentimes obtain full, administrator-level control. Such intrusions lead to financial and privacy losses as users continue to transact and use these systems unknowingly. Even worse, the presence of sizeable populations of compromised systems on computer networks imposes a substantial externality on *all* network users because these systems are leveraged in other criminal activities that negatively affect the public. In fact, a continuous influx of new compromised systems is paramount to the successful operation of botnets which are central to the criminal SPAM value chain (Levchenko et al. 2011), the execution of distributed denial-of-service attacks (Fitzgerald 2015), and conducting of click fraud campaigns (Ingram 2015).

Observing today's cybersecurity attack landscape, the current patching process for security has been less effective than desired (August et al. 2014). Many systems remain unpatched long after patches are released. HP indicates in its recent *Cyber Risk Report 2015*, that "... the majority of exploits discovered by our teams attempt to exploit older vulnerabilities. By far the most common exploit is CVE-2010-2568 which roughly accounts for a third of all discovered exploit samples" (HP Security Research 2015). According to HP, 64% of the top exploit samples in 2014 targeted vulnerabilities from 2012 and prior. OPSWAT, which collects data from software users through its security platform, finds that less than 29% of Windows operating systems in their data are up-to-date (OPSWAT 2014). Similarly, the Canadian Cyber Incident Response Centre states that patching applications and patching operating system vulnerabilities (in addition to whitelisting and access control strategies) would prevent as much as 85% of targeted security attacks (CCIRC 2014).

The growing reality is that security is not a technical problem, it's an economic one. Even though security patches are available, many users are not deploying them because it is not in their economic best interest to do so. For organizations, enterprise deployment of patches is a costly process. Extensive testing of patches in development and staging environments, roll-out of updates onto production servers, and final testing is both time consuming and resource intensive. Moreover, in aggregate there is a deluge of patches that system administrators must continuously monitor and process. For end users, the situation is regrettably similar because security patching is often not considered to be a priority. The deployment of updates and system rebooting is instead viewed as an inconvenience, particularly when users feel their own productivity is of greater concern.

Software producers have long shielded themselves from liability using well-crafted license agreements. However, the increasing breadth of software use in riskier operating environments including the critical infrastructure, biomedical products, and automobiles, comes with increased exposure to strict liability. For example, researchers have demonstrated the ability to hack into automotive systems and hijack control over brakes and steering from drivers; vulnerabilities such as these can be utilized to cause physical harm to citizens (Greenberg 2015). The security landscape is evolving in this manner, and there will be more pressure than ever on software producers to become responsible for the security of software and the ecosystem within which it runs. Positions such as the one Microsoft took in 2013 when it completely ended support for Windows XP, placing 39% of the personal computer user base at risk without security updates, will become less appealing (Jones 2013). Instead, vendors will have increased incentives to determine ways to make their products less attack worthy in the eyes of motivated hackers.

In this paper, we offer an approach on how a software vendor can substantially increase security in an incentive compatible way by encouraging improved user behavior. In particular, we argue that a vendor should differentiate its software product by pricing *patching rights*. Specifically, the vendor should charge users for the <u>right</u> to choose for themselves whether patches are installed or not installed on their systems. The status quo is that all users are endowed with patching rights, and a substantial portion of them elect not to exercise this right as a result. By charging for patching rights, users who would otherwise have elected not to patch under the status quo must now examine whether it is worth paying for a right that goes unexercised. This decision is non-trivial as the expected security losses one would incur when retaining rights and remaining unpatched depends on the security behaviors of all other users in aggregate. On the flip side, by foregoing patching rights, users will have their systems automatically patched by the vendor and pay a lesser price. In this light, users who give up patching rights cause less of a security externality and benefit from discounted software prices.

We study the impact of optimally priced patching rights for a software product on the security of the network, profitability, and the overall value of the product to the economy. While software vendors tend to be painted as not caring about security, realistically their strategic motivations are much more complex. In fact, a riskier network is not just harmful to the users who bear losses. The presence of a large unpatched user population creates disincentives for software usage which, in turn, certainly hurts profitability for the vendor. We construct a model of security where users can choose whether to purchase a software product and additionally whether to remain patched, unpatched, or have their systems automatically updated by a software vendor. By characterizing equilibrium consumer behavior in this setting, we can explore the potential benefits associated with the proper pricing of patching rights. We then examine how our insights extend to the open-source software (OSS) domain. Many OSS products are made available for free, thus we propose how a priced patching rights policy can be implemented in this domain through taxes. We compare the relative value of priced patching rights in proprietary and open-source settings, and characterize how the advancement of automated patching technologies affects the magnitude of taxing required.

2 Literature Review

This work is related to three broad areas in the literature: (i) product differentiation and market segmentation, (ii) economics of information security, and (iii) economics of open-source software. With regard to the first stream, our paper is the first to examine how beneficial segmentation in software markets can be constructed by differentiation on software patching rights. Within the second area, our paper is most closely related to a strand that studies the management of security patches. For the third area, our paper adds to a strand that examines the connection between OSS and security. We make several contributions to these areas. Our work is based on the original idea that *patching rights* should be managed. Having been first introduced qualitatively in a perspectives piece (August et al. 2014), our paper is the first to formally model and analyze the value of patching rights and clarify the impact that the pricing of patching rights has on security and the value of software products. We aim to make a significant contribution to the literature in this area where many researchers are working toward improving security and understanding why simple patch availability has not led to very secure outcomes. Importantly, the insights generated with our model have promising practical implications to the software industry. Software companies have likely not considered differentiating on patching rights, and will be interested to learn how a more profitable, more secure ecosystem can be achieved as a result.

In the following, we will detail how our paper fits with each of these areas in the literature, beginning with the first. There is a well-developed literature in economics, marketing, operations management and information systems that examines the monopolist's problem of whether to offer and how to price quality-differentiated goods. Mussa and Rosen (1978) and Maskin and Riley (1984) are foundational works that study this problem and characterize the monopolist's optimal non-linear price schedule. Moorthy (1984) generalizes this work to include nonlinear consumer preferences to demonstrate how market segments become aggregated to reduce cannibalization with the firm's own product line. Gabszewicz et al. (1986) study how income dispersion affects the monopolist's offerings, demonstrating that only the highest quality product is offered under narrow dispersion and the maximum number of qualities (and hence, segments created) is offered as dispersion becomes wider. Desai (2001) studies the product line design problem when consumers differ in their taste preferences in addition to their quality valuations. He shows that with two consumer segments (in quality valuations) and differing taste preferences, it can be in the monopolist's best interest to provide both segments with their preferred quality level. Anderson and Dana (2009) develop a general model of price discrimination to characterize the conditions under which a firm profitably offers multiple products when it can only imperfectly segment its customers. They establish that an important condition is that the percentage change in social surplus from product upgrades should be increasing in consumer willingness to pay.

Moorthy and Png (1992), Chen (2001), and August et al. (2015) study how delays can be utilized to achieve market segmentation in various contexts. Villas-Boas (2004) studies the product line design problem when marketing the differences between products is costly, in which case the product line may contract. Further, with vertically differentiated products, advertising costs lead the firm to advertise the lower quality product more intensely because lower value consumers buy only the lower quality product, and higher value consumers still derive surplus from consumption of the lower quality product; however, lower value consumers do not buy the high quality product. Debo et al. (2005) study an integrated market segmentation and production technology choice problem of a firm considering the sale of remanufactured products in the market. Netessine and Taylor (2007) study a model that combines a product line problem together with an economic order quantity (EOQ) production setting. They examine how the interactions between the number of products offered and production economies of scale, as well as product quality and inventory holding costs, affect the optimal product line. Villas-Boas (2009) studies the product line design problem when consumers have evaluation costs. This paper establishes how an expanded product line's association with higher prices may lead to fewer consumers engaging in product search. Thus, contraction of the product line can be warranted even in the absence of costs.

We study how a monopolist producer of software can expand its product line by differentiating based on the patching rights included in each offering. In particular, the monopolist can charge users for the right to choose whether or not security patches are deployed on their systems; in a similar vein, he can enforce automatic deployment of security patches to users' systems who prefer not to pay to retain this right. In our model, the effective quality of the product is inclusive of security attacks which are endogenously determined by the strategic protection behaviors employed by consumers in equilibrium. Thus, a product line based on patching rights can encourage consumers to self-select into preferable segments which can benefit endogenous security quality, generate additional profits, and even positively affect social welfare. To the best of our knowledge, our paper is the first to examine market segmentation based on patching rights. We highlight the practical implications of rights-based differentiation to the software industry, covering both commercial and open-source products.

Next, we discuss the literature that studies the management of security patches. Several papers examine the optimal *timing* of security patch release and application, along with its close connection to vulnerability disclosure. Beattie et al. (2002) characterize the optimal time to apply patches when trading off patch instability and security risk exposure. Dey et al. (2015) compare several patch application policies, varying by a measure of interest including the number of patches, time between patching, and cumulative severity of patches. Cavusoglu et al. (2008) examine the role of cost sharing and loss liability on time-driven patch management. Cavusoglu et al. (2007) and Arora et al. (2008) examine the interaction between a vendor's patch release timing and the disclosure of vulnerabilities by a social planner.

Complementing this work on timing, several papers examine user patching incentives once patches become available. August and Tunca (2006) present a base model of software purchasing and patching in the presence of negative security externalities and patching costs, and then study the impact of patching mandates, rebates, and taxes. Choi et al. (2010) study the link between users' patching incentives and the issue of vulnerability disclosure. Lahiri (2012) and Kannan et al. (2013) study various aspects of the relationship between security patches and piracy. Our work is closer in spirit to the latter group of papers which employ models with a focus on users' patching incentives. We build on this body of work and construct an originative model that includes an automated patching option for users within a game theoretical context accenting negative externalities stemming from unpatched behavior. This inclusion serves two purposes. First, it permits a characterization of the natural consumer market segmentation that arises in equilibrium as users strategically respond to security risk and expanded patching options. An understanding of equilibrium consumption and security behavior serves to inform how security enhancements should be marketed. Second, an automated patching option is the logical choice for the baseline product in a policy where patching rights are contracted, which is the focus of our work.

Our paper is broadly connected to the greater literature on the economics of information security. Png and Wang (2009) and Kannan et al. (2013) examine the interaction between protection decisions and attacker efforts. Chen et al. (2011) and August et al. (2014) demonstrate the benefits of diversification on security. Kannan and Telang (2005) and Ransbotham et al. (2012) contribute to the discussion on vulnerability disclosure by examining whether markets for vulnerabilities are helpful to measures of security and social welfare.

Which is more secure: open source or proprietary software? This question has been the center of intense debate for years. OSS security vulnerabilities can be more easily spotted by developers, some of whom may offer fixes. However, these vulnerabilities can also be more easily spotted and attacked by malicious hackers as well. On the other hand, proprietary software might have secrecy working in its favor although security through obscurity is largely considered not a good strategy. The security community has argued both sides of the nuances of these observations. Schneier (1999) contends that in cryptography the algorithm is typically open to assure correctness. Thus, public algorithms which are designed to be secure even though they are open necessarily need to be more secure than proprietary ones. In his view, OSS security should be similar since it cannot simply rely on keeping the code secret. Other cryptography experts agree that opposing claims by proprietary vendors are refutable (Diffie 2003). Schneider (2000) believes that the incentives structure for developers and hackers has created a security landscape that is not governed by bugs discoverable from opening source code - the real security problems lie elsewhere. Many experts take the view that opening the source is necessary to build more secure systems, but certainly not sufficient (Hoepman and Jacobs 2007, Wheeler 2003). Others suggest that whether open source can improve security really comes down to the underlying economic incentives of the firms, users, and hackers (Witten et al. 2001, Anderson 2002).

We contribute to the discussion of this question by comparing security measures across OSS and proprietary contexts. Our focus however lies on how the difference in pricing (vendor-optimized proprietary price and free OSS) leads to starkly different equilibrium usage and unpatched behavior, with varying security implications in turn. Using our model, we are able to examine how the efficacy of priced patching rights policies compares across source code strategies. In the case of OSS, we study how a welfare-motivated project organizer (social planner) would price (tax) these rights. Thus, our work tackles this debate from a unique perspective that focuses on security as driven by user incentives.

3 Model Description and Consumer Market Equilibrium

3.1 Model

There is a continuum of consumers whose valuations of a software product lie uniformly on $\mathcal{V} = [0, 1]$. The software is used in a network setting, thus exposing consumers to security risks associated with its use. In particular, a vulnerability can arise in the software in which case the vendor makes a security patch available to all users of the software. Because the security vulnerability can be used by malicious hackers to exploit systems, users who do not apply the security patch are at risk.

The vendor offers two options for users to protect their respective systems. In doing so, the vendor prices the software based on whether patching rights are granted to the consumer. Specifically, if a consumer elects to purchase the software and retain full patching rights, she pays the price $p \geq 0$. Having this right means she can choose whether to patch the software or not patch the product and do so according to her own preferences. If she decides to patch the software, she will incur an expected cost of patching denoted $c_p > 0$. The patching cost accounts for the money and effort that a consumer must exert in order to verify, test, and roll-out patched versions of existing systems. If she decides not to patch the software, the probability she is hit by a security attack is given by $\pi_s u$, where $\pi_s > 0$ is the probability an attack appears on the network and u is the size of the unpatched population of users. This reflects the negative security externality imposed by unpatched users of the software. If she is successfully attacked, she will incur expected security losses that are positively correlated with her valuation. That is, consumers with high valuations will suffer higher losses than consumers with lower valuations due to opportunity costs, higher criticality of data and loss of business. For simplicity, we assume that the correlation is of first order, i.e., the loss that a consumer with valuation v suffers if she is hit by an attack is $\alpha_s v$ where $\alpha_s > 0$ is a constant. The quantity $\pi_s \alpha_s$ can be considered as the *effective security loss factor*.

On the other hand, if she elects to purchase the software and relinquish patching rights, she pays the price δp, where $\delta \in [0, 1]$. In this case, the vendor retains full control over patching the software and will automatically do so to better protect the network.[1] The user incurs a cost of

[1]In interconnected networks, this is fairly easy to enforce; for example, vendors such as Adobe and Matlab enforce

automated patching, $c_a > 0$, which is associated with both inconvenience and configuration of the system to handle automatic deployment of security patches gracefully. This cost is assumed to be lower than the cost of patching, i.e., $c_a < c_p$. In general, a software vendor who releases a security patch cannot test for compatibility of the patch with every possible user system configuration. Thus, there is always some risk associated with an automatically deployed patch causing a user's system to become unstable or even crash. We denote the probability that the automated patch fails with $\pi_a > 0$. We assume that the loss associated with an automated patch deployment failure is again positively correlated with her valuation, and that this correlation is of first order, denoted as $\alpha_a > 0$. Thus, her expected loss associated with automated patching is given by $\pi_a \alpha_a v$.

Each consumer makes a decision to buy, B, or not buy, NB. Similarly, the patching decision is denoted by one of patch, P, not patch, NP, and automatically patch, AP. In order to choose P or NP, the consumer must pay the premium price p to retain patching rights. By choosing AP, the consumer delegates patching rights to the vendor for a potentially discounted price δp. The consumer action space is then given by $S = (\{B\} \times \{P, NP, AP\}) \cup (NB, NP)$. In a consumer market equilibrium, each consumer maximizes her expected utility given the equilibrium strategies for all consumers. For a given strategy profile $\sigma : \mathcal{V} \to S$, the expected utility for consumer v is given by:

$$U(v,\sigma) \triangleq \begin{cases} v - p - c_p & if \quad \sigma(v) = (B, P)\,; \\ v - p - \pi_s \alpha_s u(\sigma) v & if \quad \sigma(v) = (B, NP)\,; \\ v - \delta p - c_a - \pi_a \alpha_a v & if \quad \sigma(v) = (B, AP)\,; \\ 0 & if \quad \sigma(v) = (NB, NP)\,, \end{cases} \tag{1}$$

where

$$u(\sigma) \triangleq \int_{\mathcal{V}} \mathbb{1}_{\{\sigma(v) = (B,NP)\}}\, dv\,. \tag{2}$$

To avoid trivialities and without loss of generality, we reduce the parameter space to $c_p, c_a \in (0,1)$, $\pi_s, \pi_a \in (0,1]$, $\alpha_s, \alpha_a \in (0,\infty)$, and $\pi_a\alpha_a \in (0, 1 - c_a)$. The latter restriction, $\pi_a\alpha_a + c_a < 1$, ensures automated patching is economical.

3.2 Consumer Market Equilibrium

Before examining how patching rights should be priced, we first must characterize how consumers segment across strategies for an arbitrary set of prices in equilibrium. Complicating the situation

real-time license checks for their subscription-based offerings. While it is always possible to circumvent protections, most paying customers are unlikely to break the license agreement.

is that the level of security risk is endogenously determined by the actions of consumers, thus we first focus on understanding the effect of their strategic interactions on equilibrium behavior due to the security externality imposed by unpatched users. The consumer with valuation v selects an action that solves the following maximization problem:

$$\max_{s \in S} \quad U(v, \sigma)\,, \tag{3}$$

where the strategy profile σ is composed of σ_{-v} (which is taken as fixed) and the choice being made, i.e., $\sigma(v) = s$. We denote her optimal action that solves (3) with $s^*(v)$. Further, we denote the equilibrium strategy profile with σ^*, and it satisfies the requirement that $\sigma^*(v) = s^*(v)$ for all $v \in \mathcal{V}$.

Lemma 1 *There exists a unique equilibrium consumer strategy profile σ^* that is characterized by thresholds v_b, v_a, $v_p \in [0, 1]$. For each $v \in \mathcal{V}$, it satisfies either*

$$\sigma^*(v) = \begin{cases} (B, P) & if \quad v_p < v \leq 1\,; \\ (B, NP) & if \quad v_b < v \leq v_p\,; \\ (B, AP) & if \quad v_a < v \leq v_b\,; \\ (NB, NP) & if \quad 0 \leq v \leq v_a\,, \end{cases} \tag{4}$$

or

$$\sigma^*(v) = \begin{cases} (B, P) & if \quad v_p < v \leq 1\,; \\ (B, AP) & if \quad v_a < v \leq v_p\,; \\ (B, NP) & if \quad v_b < v \leq v_a\,; \\ (NB, NP) & if \quad 0 \leq v \leq v_b\,. \end{cases} \tag{5}$$

Lemma 1 establishes that if a population of patched consumers arises in equilibrium, it will consist of a segment of consumers with the highest valuations. These consumers prefer to shield themselves from any valuation-dependent losses seen with either remaining unpatched and bearing security losses or selecting automated patching and bearing patch instability losses. Importantly, this segment need not arise, and $v_p = 1$ in cases where the valuation-dependent losses are smaller than the patching costs. For the middle segment on the other hand, the segment of consumers who elect for automated patching and the segment of consumers who elect to remain unpatched can be ordered either way. This ordering depends on the relative strength of the losses under each strategy.

In Section A.2 of the Appendix, we present a complete characterization of the parameter conditions and thresholds for each possible consumer market structure that can arise in equilibrium. There are seven possible structures, with two of the most relevant to the current discussion having all market segments represented in equilibrium, i.e., $0 < v_b < v_a < v_p < 1$ and $0 < v_a < v_b < v_p < 1$. These two structures obtain under broad parameter conditions which are characterized in the Appendix, and as will be seen in the next section, these conditions can be satisfied under equilibrium pricing decisions.

4 Pricing Patching Rights

4.1 Proprietary Software

We turn our attention to the pricing equilibrium for standard, proprietary software. We denote the vendor's profit function by

$$\Pi(p,\delta) = p\int_{\mathcal{V}} \mathbb{1}_{\{\sigma^*(v|p,\delta)\in\{(B,NP),(B,P)\}\}}dv + \delta p\int_{\mathcal{V}} \mathbb{1}_{\{\sigma^*(v|p,\delta)=(B,AP)\}}dv\,, \tag{6}$$

noting marginal costs are assumed to be negligible for information goods. Because we are interested in determining the benefit of optimally pricing the right for a user to determine whether or not to install patches on her system, it is useful to first present a characterization of the equilibrium when this right is not priced. In this reference case, referred to throughout the paper as the *status quo*, $\delta = 1$, which is standard practice for the industry. In this case, the vendor sets a price p for use of the software by solving the following problem:

$$\begin{aligned} &\max_{p\in[0,1]} && \Pi(p,\delta) \\ &\text{s.t.} && (v_b, v_a, v_p) \text{ are given by } \sigma^*(\cdot\,|\,p,\delta), \\ & && \delta = 1. \end{aligned} \tag{7}$$

Given a price p^* that solves (7), we denote the profits associated with this optimal price by $\Pi_{SQ} \triangleq \Pi(p^*, 1)$. Since the value of automated patching options on security is most applicable under higher security risk, our study centers on a region where $\pi_s\alpha_s$ is suitably high such that patching is worthwhile. In this region, consumers will be shown to have sufficient incentives to patch and protect their systems under equilibrium pricing. This is to say that the patching threshold satisfies

$v_p < 1$, and the consumers with valuations $v \geq v_p$ will patch in equilibrium.

Lemma 2 (Status Quo) *Suppose that $\pi_s \alpha_s > \omega$ and $\delta = 1$ (i.e., when patching rights are not priced). If $c_p - \pi_a \alpha_a < c_a \leq 1 - \pi_a \alpha_a - (1 - c_p)\sqrt{1 - \pi_a \alpha_a}$, then*

$$p^* = \frac{1}{2}\left(1 - \pi_a \alpha_a - c_a\right) + \frac{4c_a^2(1 - \pi_a \alpha_a)(c_a - \frac{1}{2}(1 - \pi_a \alpha_a - c_a)(1 - 2\pi_a \alpha_a))}{(1 - \pi_a \alpha_a + c_a)^3 \pi_s \alpha_s} + K_a\,, \tag{8}$$

and σ^* is characterized by $0 < v_b < v_a < v_p < 1$ such that the lower tier of users remain unpatched and the middle tier prefers automated patching. On the other hand, if $c_a > 1 - \pi_a \alpha_a - (1 - c_p)\sqrt{1 - \pi_a \alpha_a}$, then

$$p^* = \frac{1 - c_p}{2} - \frac{2c_p^2(1 - 3c_p)}{(1 + c_p)^3 \pi_s \alpha_s} + K_b\,, \tag{9}$$

and σ^* is characterized by $0 < v_b < v_p < 1$ such that there is no user of automated patching in equilibrium.[2]

In the first part of Lemma 2, we examine the reference case under conditions where the cost of automated patching is within a moderate range.[3] An immediate observation is that when patching rights are free as in the status quo, the consumer segment whose equilibrium strategy is to use automated patching is always the middle tier. This occurs because when a user compares an automated patching strategy (B, AP) to an unpatched strategy (B, NP), the price is the same for both options, i.e., $p = \delta p$ when $\delta = 1$. Therefore, the strategy (B, AP) is preferred to (B, NP) as long as $v[\pi_s \alpha_s u(\sigma^*) - \pi_a \alpha_a] > c_a$ is satisfied. If this inequality is satisfied for any user with valuation v, it will also be satisfied for any user with a valuation higher than v. Thus, the automated patching segment of users will always form the middle tier.

This observation highlights an important potential impact of priced patching rights; if the premium charged for patching rights, $p(1 - \delta)$, is greater than c_a and the unpatched population, $u(\sigma^*)$, decreases enough in equilibrium, then the lower tier can instead be composed of users who strategically choose automated patching (see (4) in Lemma 1). In this sense, a priced patching rights policy can fundamentally change segmentation behavior in the consumer market which in turn can have a significant impact on security and profitability.

When patching rights are priced, the vendor jointly selects $(p,\ \delta)$ to maximize his profits. In this case, the premium $p(1 - \delta)$ is charged for patching rights, regardless of whether the patching

[2] In the Appendix, the existence of ω and characterization of K_a and K_b are proven. Going forward, we will represent constants that are of order $O\left(1/(\pi_s \alpha_s)^2\right)$ using the notation K and an enumerated subscript.

[3] Our focus here is on a region in which the three options, $(B, P), (B, NP)$, and (B, AP) are comparable and each is selected by some group of users in equilibrium. This is the purpose of the conditions on c_a.

rights are exercised. Alternatively, $p(1-\delta)$ can be considered the "discount" given to users who agree to have their systems automatically updated to reduce security risk on the network. When patching rights are priced in this fashion, the vendor's pricing problem is formulated as follows:

$$\begin{aligned} \max_{(p,\delta)\in[0,1]^2} \quad & \Pi(p,\delta) \\ \text{s.t.} \quad & (v_b, v_a, v_p) \text{ are given by } \sigma^*(\cdot|p,\delta). \end{aligned} \tag{10}$$

Similarly, under the optimal (p^*, δ^*) which solve (10), we denote the associated profits by $\Pi_P \triangleq \Pi(p^*, \delta^*)$.[4]

Lemma 3 (Priced Patching Rights) *Suppose that $\pi_s\alpha_s > \omega$ and patching rights are priced by the vendor.*

(i) If $c_a < \min\left[\pi_a\alpha_a - c_p, c_p(1-\pi_a\alpha_a)\right]$, then

$$p^* = \frac{1-c_p}{2} + \frac{2c_p^2(\pi_a\alpha_a)^2(3(c_a-c_p)+\pi_a\alpha_a)}{(c_a-c_p-\pi_a\alpha_a)^3\pi_s\alpha_s} + K_c, \tag{11}$$

$$\delta^* = \tilde{\delta} - \frac{4c_p^2(\pi_a\alpha_a)^2(1-c_a-\pi_a\alpha_a)(3(c_a-c_p)+\pi_a\alpha_a)}{(1-c_p)^2(c_a-c_p-\pi_a\alpha_a)^3\pi_s\alpha_s} + K_d, \tag{12}$$

and σ^* is characterized by $0 < v_a < v_b < v_p < 1$, where $\tilde{\delta} = \frac{1-\pi_a\alpha_a-c_a}{1-c_p}$, such that the lower tier of users prefer automated patching and the middle tier remains unpatched;

(ii) However, if $|\pi_a\alpha_a - c_p| < c_a < c_p(1-\pi_a\alpha_a)$, then

$$p^* = \frac{1-c_p}{2} + \frac{c_a(-1+c_a+2c_p+\pi_a\alpha_a-2c_p\pi_a\alpha_a)(c_a+c_p-c_p\pi_a\alpha_a)}{(1+c_a-\pi_a\alpha_a)^3\pi_s\alpha_s} + K_e, \tag{13}$$

$$\delta^* = \tilde{\delta} + \frac{2c_a(-1+c_a+2c_p+\pi_a\alpha_a-2c_p\pi_a\alpha_a)\left(c_a^2+(-1+\pi_a\alpha_a)\left(c_p^2+\pi_a\alpha_a-2c_p\pi_a\alpha_a\right)\right)}{(1-c_p)^2(1+c_a-\pi_a\alpha_a)^3\pi_s\alpha_s} + K_f, \tag{14}$$

and σ^* is characterized by $0 < v_b < v_a < v_p < 1$ such that the lower tier of users remains unpatched and the middle tier prefers automated patching.

Lemma 3 demonstrates that a restructuring of the consumer market can indeed be the equilibrium outcome when patching rights are priced. Specifically, if the patching costs are small such

[4] Going forward, we will use subscripts "SQ" and "P" to indicate a particular measure refers to the outcome under the status quo and under priced patching rights, respectively, for consistency.

that part (i) of Lemma 3 is satisfied, then the equilibrium patching rights are priced in a way that consumers who select automated patching in equilibrium form the lower tier of the consumer market. By (11) and (12), it is clear that the premium charged for patching rights $p^*(1-\delta^*)$ is only greater than c_a when c_p is small enough. When standard patching costs are small, the software vendor has an incentive to charge a high price for his software. One can think of it as being better software - cheap to keep fully maintained to avoid security losses. The vendor can achieve a relatively large user population, most of which chooses standard patching, even with a high price. In this case, it is necessary to give a significant discount to users in order to incentivize them to elect automated patching because of standard patching's cost effectiveness. This is reflected in (12); δ^* decreases as c_p decreases. Thus, the patching rights premium $p^*(1-\delta^*)$ becomes substantial and is an incentive-compatible option only for higher valuation users. Said differently, low valuation users will find the patching rights premium to be too large, and will choose automated patching in equilibrium.

On the other hand, when the patching rights premium is limited, the equilibrium price and discount induce a consumer market structure that more closely resembles what unfolds under the status quo. Part (ii) of Lemma 3 shows that this structure is characterized by the threshold ordering $0 < v_b < v_a < v_p < 1$, matching the threshold ordering in the first part of Lemma 2. Thus, under both the status quo and under priced patching rights, the middle tier is incentivized to select the automated patching option in equilibrium.

Next, we turn toward examining the value of priced patching rights to the software industry and the security of software networks.

Proposition 1 *For sufficiently high $\pi_s\alpha_s$, if $c_p - \pi_a\alpha_a < c_a < 1 - \pi_a\alpha_a - (1-c_p)\sqrt{1-\pi_a\alpha_a}$, then pricing patching rights can improve profits while reducing the security externality generated by unpatched users as compared to when patching rights are not priced. The percentage improvement in profitability is given by*

$$\frac{\Pi_P - \Pi_{SQ}}{\Pi_{SQ}} = \frac{(1-\pi_a\alpha_a)(c_a - c_p + \pi_a\alpha_a)^2}{\pi_a\alpha_a(1-c_a-\pi_a\alpha_a)^2} + K_g\,. \tag{15}$$

Proposition 1 highlights an important message from our study: software vendors should begin considering differentiation of their products based on patching rights. Simply providing patches for security vulnerabilities of software to users as a security strategy has not worked well in the past. In many cases, this leads to large unpatched user populations as these users determine its not in their best interest to patch. The externality they cause is detrimental to security and to the vendor's profitability. Proposition 1 formally establishes that the proper pricing of patching rights can increase profits for vendors to an extent characterized in (15) and simultaneously reduce the size of the unpatched population in the network. Thus, there are large potential economic and security benefits associated with a priced patching rights strategy, which can be an important pricing paradigm shift for the software industry.

Product differentiation is an important topic studied in economics and marketing, and the versioning of information goods has further nuanced findings (Bhargava and Choudhary 2001, 2008, Johnson and Myatt 2003). In particular, for these goods which have a negligible marginal cost of reproduction, a software vendor finds it optimal to release only one product (no versioning) when consumers heterogeneous taste for quality is uniformly distributed. In such a case, cannibalization losses outweigh differentiation benefits. In the current work, Proposition 1 demonstrates that if the versioning is instead on patching rights, a versioning strategy is once again optimal for the vendor. In this case, the software vendor can profitably benefit by increasing the price of the version with patching rights (p^*) relative to the price point under the status quo. By doing so, while concurrently decreasing the price of the version without patching rights (automated patching only) to ($\delta^* p^*$), there are several effects as consumers strategically respond. First, a higher p^* puts pressure on any user who would be unpatched under the status quo to reconsider the trade-off. Under the status quo equilibrium consumer market structure (i.e., $0 < v_b < v_a < v_p < 1$), the unpatched users form the lower tier of the consumer market (those with valuations between $[v_b, v_a]$). Because patching rights are endowed to all users under the status quo, these users remain unpatched and contribute to a larger security externality on the network. Under priced patching rights, a higher p^* makes it now more expensive to remain in the population as an unpatched user causing this externality. Second, given the new equilibrium prices, it becomes relatively cheaper to opt for automated patching at a discount of $p^*(1 - \delta^*)$. This provides additional incentives to encourage better security behaviors. On the other hand, a higher price can be detrimental to usage and associated revenues, and a reduced unpatched population can create incentives for users who were patching under the status quo to now remain unpatched.

The net impact of these effects depends on which consumer market structure is induced by the vendor's new prices. As Lemma 3 demonstrates, the vendor may induce a segmentation characterization of either $0 < v_a < v_b < v_p < 1$ or $0 < v_b < v_a < v_p < 1$. We begin by discussing the latter structure since it matches the status quo. In equilibrium under patched pricing rights, v_b increases and v_a decreases relative to the status quo. Thus, the size of the unpatched population (i.e., $u = v_a - v_b$) shrinks as it is compressed on both ends. However, v_p increases because of the patching rights premium. In aggregate, the vendor is able to increase profitability by charging a premium to high tier consumers (valuations in $[v_p, 1]$) who are willing to pay the premium to protect their valuations from incurrence of either unpatched security losses or automated patching instability losses, and low tier consumers (valuations in $[v_b, v_a]$ are willing to pay the premium because of the smaller security externality that is associated with a smaller equilibrium unpatched user population.

For the former case in which the vendor sets prices such that the equilibrium consumer market structure characterization takes the form $0 < v_a < v_b < v_p < 1$, there is a restructuring in the consumer market segments (see the discussion following Lemma 3). It is in the vendor's best interest to have a relatively large patching rights premium in this region which makes the retaining of patching rights only incentive compatible for the higher valuation users. Low valuation users respond to a substantial discount by forgoing patching rights and switching to automated patching. Because low valuation users tend to be the ones with reduced incentives to patch and protect themselves, the market segmentation that occurs also leads to a smaller unpatched population and less resultant security risk. In a similar spirit to the discussion above, this is profitable to the vendor as it is able to raise prices due to greater security and high valuations users' willingness to pay to retain patching rights.

By characterizing the percentage improvement in profitability associated with a priced patching rights strategy in (15), we can highlight the type of market characteristics where efforts for a vendor to reexamine patching rights is more fruitful. In particular, the relative improvement in profitability is increasing in c_a and decreasing in c_p. As the cost of automated patching increases through the relevant region (see Proposition 1), under the status quo the vendor necessarily reduces the software's price to make the automated patching option continue to be affordable. This is important because it prevents a significant loss in users resulting from higher security risk that can arise if the automated patching option becomes too costly. On the other hand, under priced patching rights the vendor can achieve a similar effect by strategically adjusting the discount targeted to the users of the automated patching option rather than the entire user population. When the cost of standard patching (c_p) decreases, the vendor achieves a relatively larger increase in profits. In this case, the premium charged to users who elect to retain patching rights can be increased as patching costs become lower.

One interesting implication of our model concerns a comparison of prices under the status quo and under optimally priced patching rights. One might expect that if p^*_{SQ} is the price under the status quo, then $\delta^* p^* < p^*_{SQ} < p^*$ is satisfied under equilibrium when patching rights are priced. That is, users who want to retain patching rights pay a premium and users who opt for automated patching receive a discount relative to the status quo. However, in the following proposition we demonstrate that the vendor may strategically raise both prices in equilibrium, in comparison to status quo pricing.

Proposition 2 *For sufficiently high* $\pi_s\alpha_s$ *and* $c_p - \pi_a\alpha_a < c_a < 1 - \pi_a\alpha_a - (1-c_p)\sqrt{1-\pi_a\alpha_a}$*, when either*

(i) $c_a < \min\left[\pi_a\alpha_a - c_p, c_p(1-\pi_a\alpha_a), \frac{1}{3-2\pi_a\alpha_a} - \pi_a\alpha_a\right]$*, or*

(ii) $|\pi_a\alpha_a - c_p| < c_a < \min\left[c_p(1-\pi_a\alpha_a), \frac{(1-2c_p)(1-\pi_a\alpha_a)}{5-4\pi_a\alpha_a}\right]$*,*

the vendor prices patching rights such that both p^* *and* $\delta^* p^*$ *are higher than the common price,* p^*_{SQ}*, when patching rights are endowed to all users.*

Not only does the endowment of patching rights lead to excessive security risk due to poor patching behavior, it also fails to reflect the value of security provision being offered by the vendor. Vendors who create better, more secure solutions for their customers should be able to harvest some of that value creation via increased prices. Proposition 2 highlights this important point by characterizing broad regions where the vendor increases the price of both options above the single price offered in the case of the status quo. This occurs for a lower level of automated patching costs (c_a), and the reason both prices increase is twofold. First, users who prefer to retain patching rights are willing to pay more for smaller unpatched populations (i.e., reduced security risk) and control over their own patching process. Second, the value associated with cost-efficient and more secure, automated patching options is more readily harvested when users of this option are ungrouped from users who choose not to patch under the status quo. The pricing of patching rights helps to enable this separation. Thus, when a vendor differentiates in this manner based on "rights," he can simultaneously increase prices, encourage more secure behaviors, and generate higher profits. The outcome under this business strategy is noteworthy because it is starkly different than one in which security protections are sold and those who opt out are both unprotected and cause a larger security externality.

Proposition 2 suggests that usage may become more restricted with priced patching rights. Moreover, it is unclear how specific costs associated with security would be affected as consumers strategically adapt their usage and protection decisions. Proposition 1 demonstrates that priced patching rights can reduce the size of the unpatched population relative to the status quo which in turn implies the risk associated with security attacks decreases. However, the magnitude of losses associated with these attacks critically depends on who actually bears them as they are valuation-dependent and consumers' equilibrium strategies will shift when patching rights are priced. We denote the expected losses associated with security attacks stemming from the unpatched population $u(\sigma^*)$ with

$$SL \triangleq \int_{\mathcal{V}} \mathbb{1}_{\{\sigma^*(v)=(B,NP)\}} \pi_s \alpha_s u(\sigma^*) v dv \,. \tag{16}$$

In a similar fashion, we denote the expected losses associated with configuration and instability of automated patching with

$$AL \triangleq \int_{\mathcal{V}} \mathbb{1}_{\{\sigma^*(v)=(B,AP)\}} (c_a + \pi_a \alpha_a v dv) \,, \tag{17}$$

and the total costs associated with standard patching with

$$PL \triangleq \int_{\mathcal{V}} \mathbb{1}_{\{\sigma^*(v)=(B,P)\}} c_p dv \,. \tag{18}$$

The net impact of consumers changing their patching strategies (standard patching, remaining unpatched, electing for automated patching) on these security-related costs is unclear. In order to examine these concerns in aggregate, we also define total security-related costs as the sum of these three components:

$$L \triangleq SL + AL + PL \,, \tag{19}$$

in which case social welfare can be expressed as

$$W \triangleq \int_{\mathcal{V}} \mathbb{1}_{\{\sigma^*(v)\in\{(B,NP),(B,AP),(B,P)\}\}} v dv - L. \tag{20}$$

In the following proposition, we establish that when automated patching costs are not too large, the pricing of patching rights can in totality have a negative effect on social welfare. This result is interesting in that both losses associated with security attacks and total costs associated with standard patching can be shown to decrease when patching rights are patched, and yet priced patching rights can still be detrimental from a welfare perspective.

Proposition 3 *For sufficiently high $\pi_s\alpha_s$, if $c_p - \pi_a\alpha_a < c_a < 1 - \pi_a\alpha_a - (1-c_p)\sqrt{1-\pi_a\alpha_a}$, then priced patching rights can either decrease or increase security attack losses, but leads to a small decrease in social welfare. Technically, $PL_P < PL_{SQ}$, $AL_P > AL_{SQ}$, $W_P < W_{SQ}$ and*

(i) if $c_a < \min\left[\pi_a\alpha_a - c_p, c_p(1-\pi_a\alpha_a)\right]$, and

$$\frac{4c_p^2\pi_a\alpha_a}{-c_a + c_p + \pi_a\alpha_a} - \frac{(c_a(2-\pi_a\alpha_a) + \pi_a\alpha_a(1-\pi_a\alpha_a))^2}{(1+c_a-\pi_a\alpha_a)(1-\pi_a\alpha_a)} > 0\,, \tag{21}$$

then $SL_P \geq SL_{SQ}$;

(ii) otherwise, $SL_P < SL_{SQ}$.

The parameter region in Proposition 3 corresponds to c_a being relatively lower and satisfying the conditions of Lemmas 2 (first part) and 3. Recalling that under priced patching rights, the consumer market structure can be characterized by either $0 < v_b < v_a < v_p < 1$ or $0 < v_a < v_b < v_p < 1$ in equilibrium, we first examine the former case where the consumer market structure matches the characterization under the status quo. By pricing patching rights, the vendor will induce an expansion of the consumer segment that elects for automated patching on both sides. That is, some unpatched users as well as some standard patching users under the status quo will now choose automated patching. Additionally, some unpatched users are now out of the market due to the increase in the price p^* associated with retained patching rights (technically, v_b increases). Therefore, losses associated with unpatched security attacks and costs associated with standard patching are both lower in comparison to the status quo, i.e., $SL_P < SL_{SQ}$ and $PL_P < PL_{SQ}$.

However, the expansion of the consumer segment choosing (B, AP) turns out to be costly. In particular, because consumers have the opportunity to relinquish patching rights to save the premium $(1-\delta^*)p^*$, the consumers that make up the expansion of this segment may incur greater security investments and system instability losses in order to avoid paying this premium. For example, at the higher end of the valuation space, a consumer may have incurred only c_p under status quo pricing but when incentivized to shift to automated patching because of the discount, she now incurs a security cost of $c_a + \pi_a\alpha_a v$ which is valuation-dependent and can exceed c_p. A similar increase in costs can arise at the lower end as consumers shift from losses associated with security attacks to investments and instability losses associated with automated patching. Proposition 3 formally establishes that the decrease in usage and increased aggregate costs incurred related to automated patching ultimately outweigh the reduction in security attack losses and standard patching costs from a welfare perspective. From a software vendor's perspective, the ability to market their product offerings as geared to reduce security risk and attack losses while increasing profits is enticing, and having awareness of the impact on welfare can help shape these initiatives. Encouragingly, we also characterize several regions where social welfare is positively impacted by priced patching rights as well in Proposition 4 and the discussion of Figure 1.

But first turning attention to the case where the vendor's pricing behavior induces a restructuring of segmentation in the consumer market to $0 < v_a < v_b < v_p < 1$, we find the outcome is similar but has some nuanced differences. In this case, the consumers whose equilibrium strategy is to retain patching rights but not patch (users with valuations between v_b and v_p) have higher valuations than those preferring this strategy under the status quo case. Thus, even though the size of the unpatched population, $u(\sigma^*)$, decreases under priced patching rights, the higher valuations of the consumers exhibiting the risky, unpatched behavior can result in them incurring higher losses when bearing security attacks. It hinges on whether $u(\sigma^*)$ decreases sufficiently to offset the higher valuations of the risky population. In part (i) of Proposition 3, the conditions required for the restructured consumer market as laid out in Lemma 3 appear. Further, (21) provides the condition whereupon security attack losses are, in fact, higher under priced patching rights, despite the reduction in unpatched usage. One can think of this outcome as characterized by fewer attacks but on higher value targets leading to greater losses in equilibrium. This condition tends to be satisfied as the likelihood of automated patch instability increases which provides more incentive for consumers to remain unpatched instead. With the potential of security attack losses to also increase, welfare is even further suppressed compared to the status quo.

Next, we study the case where automated patching costs are at a level large enough that an automated patching segment is absent under the status quo but small enough that this segment arises when patching rights are priced (see the second part of Lemma 2 and Lemma 3).

Proposition 4 *For sufficiently high $\pi_s\alpha_s$, if $1-\pi_a\alpha_a-(1-c_p)\sqrt{1-\pi_a\alpha_a} \leq c_a < c_p(1-\pi_a\alpha_a)$, then priced patching rights leads to decreased security attack losses and an increase in social welfare. Technically, $SL_P < SL_{SQ}$, $PL_P < PL_{SQ}$, $AL_P > AL_{SQ}$, and $W_P > W_{SQ}$.*

Proposition 4 examines a higher cost of automated patching in which case, under status quo pricing, the consumer market equilibrium is characterized by $0 < v_b < v_p < 1$ (Lemma 2). One can think of this as a context where automated patching technology is inferior and users elect not to use it in equilibrium. This behavior can result in a large unpatched population and substantial security risk, causing many potential consumers to prefer not to be users of the product. Thus, the value of a priced patching rights policy can be lucrative if it provides sufficient incentives to reduce unpatched behavior and expand usage. Under optimally priced patching rights, users who were unpatched under the status quo are incentivized by a discount to use the automated patching option. In that automated patching is an inferior technology in this context, these users may bear greater costs and instability losses associated with automated patching in exchange for receiving this discount. These greater costs are detrimental to welfare.

On the other hand, because the unpatched population is significantly reduced, losses associated with security attacks are lower ($SL_P < SL_{SQ}$). Moreover, because the vendor makes the automated patching available at a discount, usage in the market for the software expands. In fact, when the loss factor on automated patching technology ($\pi_a \alpha_a$) is at the high end of the focal region, both the price of the product with patching rights (p^*) and without ($\delta^* p^*$) can be lower than the price under the status quo (p^*_{SQ}). Thus, usage in the market can expand substantially, and the additional surplus generated from these consumers who were non-users under the status quo helps to benefit welfare. Proposition 4 establishes that the net effect of these factors is positive, and priced patching rights can have a positive influence on social welfare.

While the pricing of patching rights is quite effective at reducing unpatched populations and losses associated with security attacks, Propositions 3 and 4 demonstrate that its impact of welfare can be mixed when security risk is sufficiently high. In Figure 1, we examine the impact on welfare as the security loss factor becomes lower under smaller automated patching costs. As can be seen in panel (a) of Figure 1, a priced patching rights strategy can also be beneficial to both vendor profits and social welfare relative to the status quo strategy as $\pi_s \alpha_s$ decreases. Under the status quo, the consumer market equilibrium is characterized by $0 < v_b < v_a < v_p < 1$ throughout panel (a). However, two different consumer market structures are represented under priced patching rights. To the right of the discontinuity, the characterization is the same, while to the left of the discontinuity (hence lower $\pi_s \alpha_s$), the structure becomes $0 < v_a < v_b < 1$. In other words, as $\pi_s \alpha_s$ decreases, patching rights are priced in a way that significantly restructures the equilibrium consumer strategies in comparison to the status quo; consumers with high valuations retain patching rights but choose to remain unpatched and consumers with lower valuations forgo rights and either shift to automated patching or exit the market.

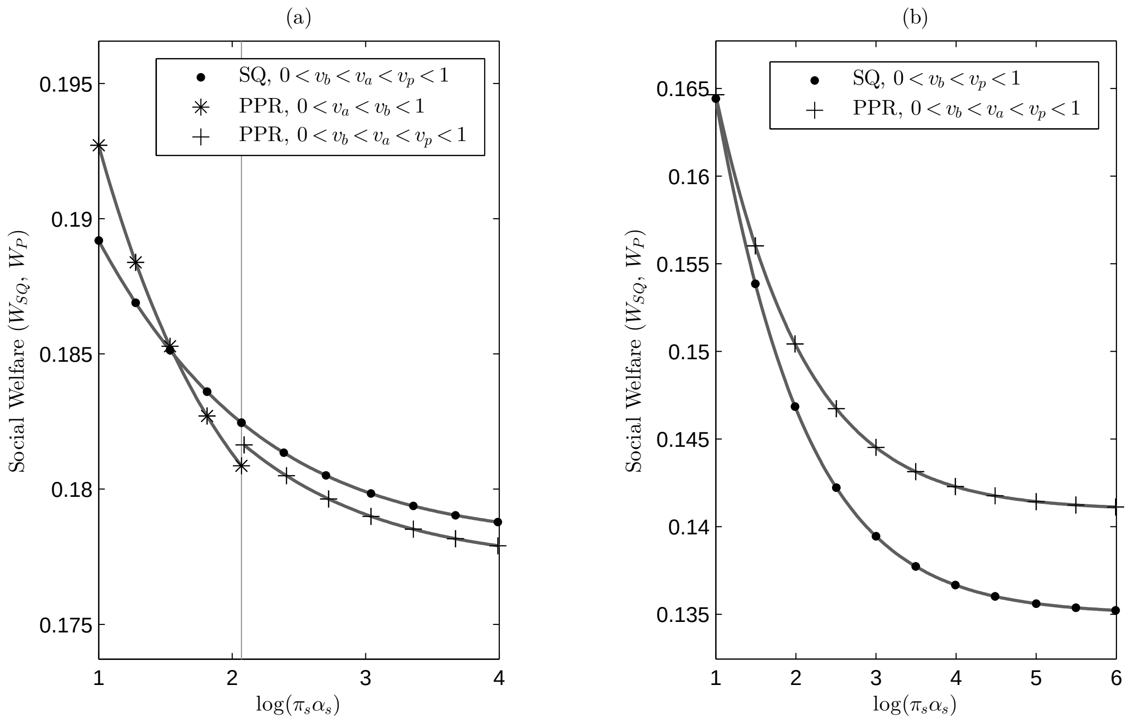

Figure 1: Impact of a priced patching rights (PPR) strategy on social welfare for varying effective security risk, compared to the status quo (SQ). In panel (a), $c_a = 0.1$, and in panel (b), $c_a = 0.2$. The common parameter values are $\alpha_a = 3.5$, $\pi_a = 0.1$, and $c_p = 0.4$.

What is most interesting about this reshuffling is that the consumers who were causing the security risk no longer do so and, as a result, the consumers who were incurring standard patching costs to shield themselves from the security risk also no longer need to do so. In this sense, a priced patching rights strategy not only reduces security risk, it enables high valuation users to avoid incurring what are typically large patching costs associated with standard patching processes that have consistently been a financial burden on organizations. The net result of priced patching rights is that total costs related to automated patching increase (the automated patching population expands), costs associated with standard patching disappear (patching burden is relieved), and security attack losses stemming from unpatched usage is reduced (significant reduction in the size of the unpatched population). As a result, social welfare increases under priced patching rights in comparison to the status quo as the security loss factor decreases out of region covered by Proposition 3.

Panel (b) of Figure 1 illustrates the finding from Proposition 4 that social welfare increases under priced patching rights for a high security loss factor. Moreover, the benefits to welfare also extend to a lower range of security losses which is depicted as well. In summary, the pricing of patching rights presents an opportunity for vendors of proprietary software to not only improve profits, but also improve welfare by decreasing the magnitude of the externality generated by unpatched usage, even to the degree that the patching burden can be relieved. In the next section, we examine how automated patching and the management of patching rights interact with open-source software which is commonly made available free of charge.

4.2 Open-Source Software

With proprietary software, the combination of an automated patching option being available and the vendor's equilibrium pricing behavior tend to together help limit the size of the unpatched population that develops, even under status quo pricing. However, for open-source software (OSS) that is freely available, an automated patching option simply being available has a quite different impact on equilibrium outcomes. One might think that because a large unpatched population arises in the absence of a price, an automated patching option would help to reduce this unpatched population drastically. However, as we will see, the unpatched population that results even with the availability of automated patching can be substantial. Therefore, a policy that effectively prices patching rights in the open-source domain has significant potential.

In the following proposition, we examine how offering an automated patching option affects the size of the unpatched population in equilibrium. To do so, we compare it to an alternative scenario in which the consumer strategy set is restricted to $\tilde{S} = (\{B\} \times \{P, NP\}) \cup (NB, NP)$ thus excluding AP. We denote the equilibrium size of the unpatched population in this case with $\tilde{u}(\sigma^*)$. Our focus is on the most relevant parameter regime where the total security costs associated with automated patching ($\pi_a\alpha_a,\ c_a$) are reasonably close in magnitude to the standard patching cost (c_p).

Proposition 5 *Suppose $c_p - c_a < \pi_a\alpha_a < 1 - c_a/c_p$. Then, the inclusion of an automated patching option for consumers has the following impact on the size of the unpatched population in equilibrium, dependent upon the level of the effective security loss factor:*

(i) if $\pi_s\alpha_s \leq \frac{c_p(\pi_a\alpha_a)^2}{(c_p-c_a)^2}$, then $u(\sigma^*) = \tilde{u}(\sigma^*)$;

(ii) if $\frac{c_p(\pi_a\alpha_a)^2}{(c_p-c_a)^2} < \pi_s\alpha_s < \frac{1-\pi_a\alpha_a}{c_a}$, then $u(\sigma^*) < \tilde{u}(\sigma^*)$;

(iii) if $\pi_s\alpha_s \geq \frac{1-\pi_a\alpha_a}{c_a}$, then $u(\sigma^*) = \tilde{u}(\sigma^*)$.

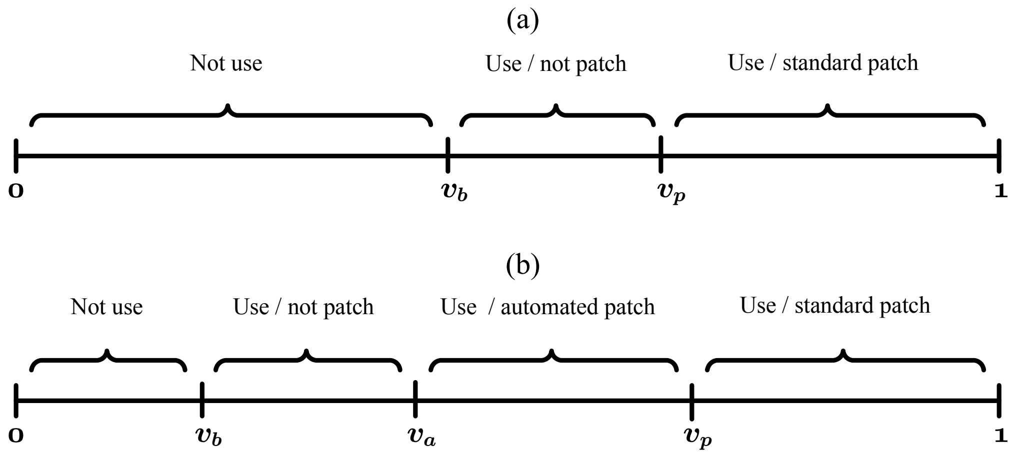

Figure 2: Equilibrium consumer market structure illustration for open-source software under a high effective security loss factor. Panel (a) depicts the structure when automated patching is not available, and panel (b) depicts the structure when it is available.

Proposition 5 makes an important statement about the role of automated patching as it relates to the security of OSS: Even with some consumers choosing to use automated patching in equilibrium, the size of the unpatched population may simply remain unchanged. Proposition 5 establishes this behavior occurs under both a low and high effective security loss factor. Only within a medium range of security losses can an automated patching lead to a reduction in the size of the unpatched population. Perhaps most unsettling is part (iii) of Proposition 5. One would hope that the beneficial impact on security of an automated patching option would be highest when the effective security losses are also high. This is not the case, and, in fact, the unpatched population is exactly the same size with or without an automated patching option.

To understand why, it is useful to describe how users segment in equilibrium. When an automated patching option is not available, users with the highest valuations still prefer the standard patching option because they are unwilling to bear valuation-dependent losses, i.e., $\pi_s \alpha_s u(\sigma^*) v$ by remaining unpatched. When OSS has zero price, all consumers with positive valuations would prefer to use the software but cannot because of the security losses associated with a large unpatched population. Therefore, consumers with lower valuations begin to enter until the security externality becomes large enough that no other consumer elects to use. The equilibrium consumer market structure is illustrated in panel (a) of Figure 2. Notably, under a high effective security loss factor, there can be a significant fraction of would-be users out of the market because of the unpatched users (those with valuations between v_b and v_p) causing risk.

When an automated patching option becomes available, users with the highest valuations still prefer the standard patching option because automated patching is also associated with valuation-dependent losses, i.e., $\pi_a \alpha_a v$. For users with moderate valuations, the trade-off shifts in favor of automated patching because they have lower value-at-risk which does not justify the higher costs associated with standard patching (recall $c_p > c_a$) to fully protect their valuations. For even lower valuation users, they will choose to remain unpatched, not even being willing to incur the cost of automated patching, c_a. This lowest segment of consumers faces the same trade-off as described in the case without automated patching - these users will continue to enter until there is a sufficiently large unpatched population that the next marginal user prefers not to use. Thus, the existence of an automated patching option only serves to shift the risky usage down the valuation space, which is depicted in panel (b) of Figure 2. What is important is that these consumers must be willing to use the software in the face of some risk, and it is precisely the lack of a price which creates a large potential user population. Therefore, the actual impact of an automated patching option for OSS is for market expansion. More consumers can become users and separate across protected forms, creating the opportunity for additional consumers (who would have been non-users without automated patching) to enter into the market. However, these additional users who enter and do not patch continue to cause an equivalent security externality on the network.

Part (ii) of Proposition 5 demonstrates that when the effective security loss factor is moderate, the size of the unpatched population can shrink when an automated patching option is available. Building on the preceding discussion, what changes in this case is that the consumer market gets covered and there is no more room to expand risky usage at the lower end of the valuation space. Thus, OSS products with moderate effective security losses that are necessarily in widespread use can benefit from an automated patching option by effecting an even further reduction in security risk. Finally, part (i) of Proposition 5 identifies conditions under which the security risk is sufficiently low such that automated patching is not a viable option, in which case there is no difference between the two scenarios.

From the preceding discussion, we see that OSS may have a sizable mass of unpatched users even when an automated patching option is offered. We next investigate the value of a policy analogous to priced patching rights in the open-source domain. In particular, we study a tax on patching rights set by a social planner to help mitigate the negative externality associated with unpatched usage. We begin by characterizing the equilibrium consumer market structures that emerge in both the open-source status quo and under the prospective tax policy, focusing on the case where security losses are appreciable and the use of automated patching solutions is incentive compatible.

Lemma 4 (OSS, Status Quo) *Suppose that* $\pi_s \alpha_s > \omega$*. If* $c_p - \pi_a \alpha_a < c_a < c_p(1 - \pi_a \alpha_a)$*, then* σ^* *is characterized by* $0 < v_b < v_a < v_p < 1$ *such that the lower tier of users remain unpatched and the middle tier prefers automated patching. If* $c_a \le c_p - \pi_a \alpha_a$*, then* σ^* *is characterized by* $0 < v_b < v_a < 1$ *such that no consumer elects for standard patching in equilibrium.*

A comparison of equilibrium consumption under the status quo across proprietary and OSS cases is revealing. Examining Lemmas 2 and 4 where the market outcome is characterized by $0 < v_b < v_a < v_p < 1$, it becomes clear that the region of the parameter space in which we observe automated patching is larger in an OSS setting than in a proprietary one. In this sense, we are currently more likely to see consumers choosing automated patching options with OSS relative to proprietary software, across software classes; this behavior is good for security. On the flip side, there will also be a relatively larger mass of unpatched users in the OSS case, which is detrimental to security. Thus, it is useful to examine how the taxing of patching rights can mitigate this downside.

Analogous to a vendor's pricing of patching rights for proprietary software, we study a government's taxing of patching rights for OSS. As a way to decrease the unpatched population size and as a result increase social welfare, the government may charge a tax ($\tau > 0$) on patching rights. For a given strategy profile $\sigma : \mathcal{V} \to S$, the expected utility for consumer v is then given by:

$$U(v,\sigma) = \begin{cases} v - \tau - c_p & if \quad \sigma(v) = (B,P)\,; \\ v - \tau - \pi_s \alpha_s u(\sigma) v & if \quad \sigma(v) = (B,NP)\,; \\ v - c_a - \pi_a \alpha_a v & if \quad \sigma(v) = (B,AP)\,; \\ 0 & if \quad \sigma(v) = (NB,NP)\,. \end{cases} \tag{22}$$

Next, we provide a characterization of both the equilibrium tax set by the government and the equilibrium consumer market structure that is induced by the tax.

Lemma 5 (OSS, Taxed Patching Rights) *Suppose that* $\pi_s \alpha_s > \omega$ *and patching rights are taxed. Then,*

(i) if $c_p - \pi_a \alpha_a < c_a < c_p(1 - \pi_a \alpha_a)$*, then*

$$\tau^* = \frac{\pi_a \alpha_a}{\pi_s \alpha_s} + K_h\,, \tag{23}$$

and σ^* *is characterized by* $0 < v_b < v_a < v_p < 1$*.*

(ii) if $\frac{(\pi_a\alpha_a)^2}{1-\pi_a\alpha_a} < c_a \leq c_p - \pi_a\alpha_a$*, then*

$$\tau^* = \frac{c_a}{2(1-\pi_a\alpha_a)} - \frac{1-3\pi_a\alpha_a}{16(1-\pi_a\alpha_a)\pi_s\alpha_s} + K_i\,, \tag{24}$$

and σ^* *is characterized by* $0 < v_b < v_a < 1$*.*

(iii) if $c_a \leq c_p - \pi_a\alpha_a$ *and* $c_a \leq \frac{(\pi_a\alpha_a)^2}{1-\pi_a\alpha_a}$*, then*

$$\tau^* = \frac{c_a + \pi_a\alpha_a}{2} + \frac{(c_a - 3\pi_a\alpha_a)(c_a + \pi_a\alpha_a)}{16\pi_s\alpha_s} + K_j\,, \tag{25}$$

and σ^* *is characterized by* $0 < v_a < v_b < 1$*.*

First, we examine part (i) of Lemma 5 where c_a is within an intermediate range such that both standard patching and automated patching populations are present in equilibrium. In this region, an optimally configured tax reduces the size of the unpatched population in such a way that the consumer indifferent between automated patching and remaining unpatched under the status quo now switches to being unpatched under the tax. In other words, the threshold v_a under the optimal tax is higher than its counterpart in the status quo. Thus, the purchasing threshold v_b (i.e., the consumer indifferent between being an unpatched user and a non-user) also moves relatively even higher in comparison to the status quo due to the tax imposition. The net effect of both thresholds increasing in this manner yields a smaller unpatched population in equilibrium. Moreover, the tax induces some users who are engaged in standard patching practices under the status quo to forgo their patching rights. This characteristic can be seen in (23); the optimal tax τ^* increases as $\pi_a\alpha_a$ increases because a social planner needs to further incentivize users of standard patching to adopt automated patching solutions when these solutions are associated with increased instability. The planner essentially achieves this by increasing the cost of patching rights to these users. Although some consumers switch away from automated patching in the status quo to being unpatched as discussed above, the movement of consumers who elect for standard patching under the status quo toward automated patching yields a larger population of users of automated patching in aggregate under the optimal tax. In the following proposition, we formally state these findings.

Proposition 6 *For sufficiently high* $\pi_s\alpha_s$ *and* $c_p - \pi_a\alpha_a < c_a < c_p(1-\pi_a\alpha_a)$*, optimally taxed patching rights decrease the size of the unpatched population by* $\frac{\pi_a\alpha_a(1-\pi_a\alpha_a)}{c_a(\pi_s\alpha_s)^2} + J_a$ *such that* $SL_P < SL_{SQ}$*, increase the size of the automated patching population by* $\frac{1}{\pi_s\alpha_s} + K_k$ *such that* $AL_P > AL_{SQ}$*, and*

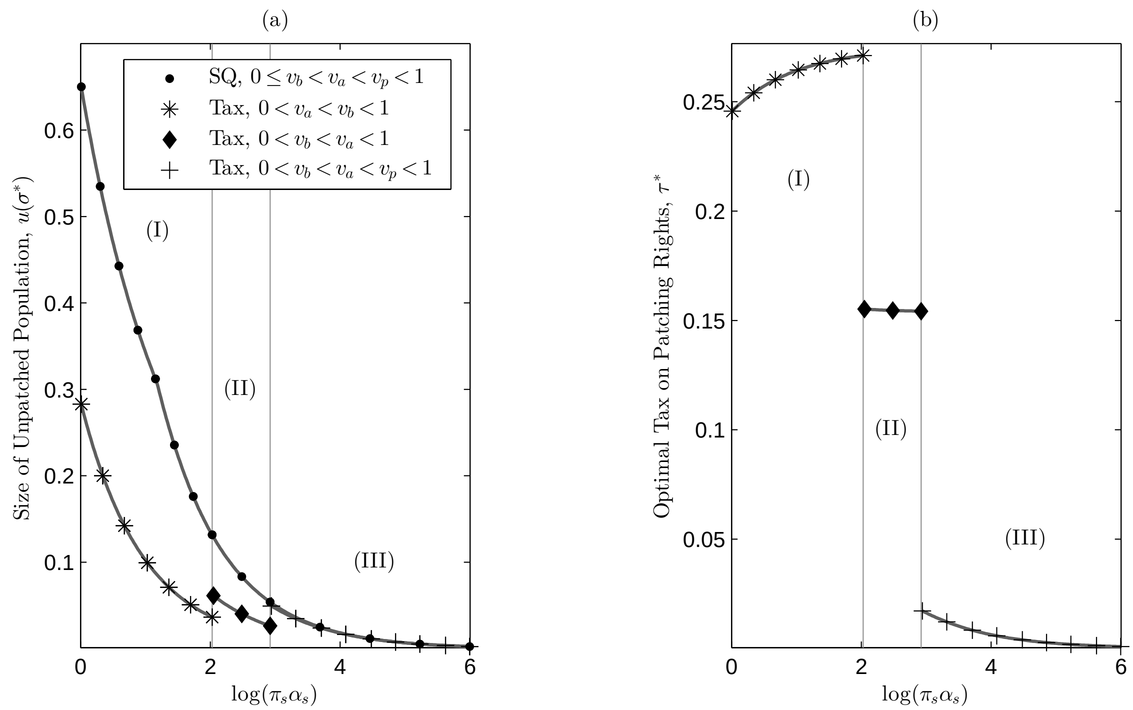

Figure 3: How the optimal tax on patching rights and associated unpatched usage in equilibrium are influenced by the effective security loss factor. Panel (a) illustrates the size of the unpatched population under both the status quo and in the presence of the tax. The optimal tax is depicted in panel (b). The parameter values are $\alpha_a = 3.5$, $\pi_a = 0.1$, $c_p = 0.5$, and $c_a = 0.2$.

increase social welfare by $\frac{\pi_a \alpha_a}{2(\pi_s \alpha_s)^2} + J_b$.[5]

By Lemmas 4 and 5, the consumer market structure induced in equilibrium under the conditions of Proposition 6 is $0 < v_b < v_a < v_p < 1$ under both the status quo and optimally configured tax. Proposition 6 demonstrates that as the effective security loss factor increases, the tax has a relatively larger impact on increasing automated patching behavior in comparison to reducing unpatched behavior. Notably, as the effective security loss factor grows quite large, users have a significant incentive not to remain unpatched even under the status quo which limits the marginal benefit of a tax. However, when the effective security loss factor has a moderate to moderately high magnitude, a tax can have an even greater impact on security risk.

Figure 3 illustrates both the equilibrium unpatched population size, $u(\sigma^*)$, and the optimal tax, τ^*, as a function of the effective security loss factor. First, we discuss the status quo. In panel (a), the upper curve represents the size of the unpatched population in equilibrium under the status quo. As $\pi_s \alpha_s$ initially increases, unpatched users with higher valuations switch to automated

[5]Similarly, we will represent constants that are of order $O\left(1/(\pi_s \alpha_s)^3\right)$ using the notation J and an enumerated subscript.

patching in order to bear relatively lower security risk. As $\pi_s \alpha_s$ increases further, unpatched users with lower valuations begin to drop out of the market, and the size of the unpatched population shrinks as is depicted. The impact of a tax on patching rights on the unpatched population is reflected by the lower curve in panel (a). Starting from the right-hand portion, Region (III) of panel (a) is consistent with Proposition 6 and illustrates how all consumer market segments are represented both under the status quo and under taxed patching rights. Panel (b) demonstrates how the optimal tax is a modest one and leads to a modest reduction in unpatched usage as seen in panel (a). As $\pi_s \alpha_s$ decreases into Region (II) and then (I), Figure 3 demonstrates how a much more significant tax is required to address mis-aligned incentives and induce a substantial reduction in unpatched usage.

Regions (I) and (II) illustrate what can happen when the effective security loss factor is not too high. Despite the consumer market structure being characterized by $0 < v_b < v_a < v_p < 1$ under the status quo, the optimal tax in these regions essentially precludes the existence of a segment of consumers who prefer standard patching in equilibrium. To see why, we begin by discussing Region (I) where the optimal tax induces the structure $0 < v_a < v_b < 1$. As can be seen in panel (b) of Figure 3, the optimal tax is set at a high level (in fact, higher than c_a) to provide incentives for consumers to forgo patching rights. In response, all consumers who were unpatched under the status quo either exit the market or choose the automated patching option. However, because of the large reduction in the unpatched population, the security risk is low and consumers with higher valuations who would be patching under the status quo now find it preferable to remain unpatched and bear the low, expected security losses in equilibrium. These consumers pay the tax to retain patching rights but need not exercise these rights. Instead, they are in spirit paying the tax to reduce security risk and hence the costly burden of standard patching processes. Notably, as panel (b) indicates, as the effective security loss factor increases through Region (I), a higher tax is needed to reduce unpatched usage by low valuation users and achieve these effects.

However, examining Region (II) in both panels of Figure 3, at some point the tax required is quite high and becomes too detrimental to total software usage in equilibrium; social welfare can be further improved by a different strategy here. In particular, in that with higher potential security risk, high valuation consumers prefer not to be exposed to higher valuation-dependent losses, more surplus would be created if a planner expands usage in the market to lower valuation consumers and provides incentives for high valuation users to switch to the automated patching option. In this case, a lower tax optimally expands usage and benefits welfare, while still limiting (albeit, to a lesser extent) the amount of unpatched behavior and associated expected security losses. In equilibrium, the consumer market structure is characterized by $0 < v_b < v_a < 1$; together, the security risk associated with expanded usage and the tax on patching rights are both sufficiently large that high valuation consumers prefer neither to pay the tax nor risk security losses. Instead, they elect for automated patching. In fact, the highest valuation consumers (who were incurring cost c_p under the status quo by standard patching) now incur cost $c_a + \pi_a \alpha_a v > c_p$ under automated patching but do not pay the tax τ.

As we discussed above, the value of taxing patching rights diminishes in security risk because it becomes more incentive compatible for users to choose patching and automated patching options. Thus, as the effective security loss factor increases even further, from a welfare perspective it is preferable for high valuation users to maintain patching rights and patch to prevent large security losses. In this case, a planner should set a small tax to encourage these users to retain rights and patch while also providing a modest disincentive for low valuation users to remain unpatched. The tax and its impact on an already smaller unpatched population is illustrated in Region (III) of Figure 3.

Having discussed parameter regions that correspond more closely to today's computing environment, we turn our attention to a region likely to unfold in the future when automated patching technology improves and becomes a less costly endeavor from the perspective of users. In Figure 4, we depict this case and illustrate how a larger tax on patching rights is utilized to significantly reduce unpatched usage in equilibrium. The following proposition formalizes our findings for a low cost of automated patching.

Proposition 7 *For sufficiently high $\pi_s \alpha_s$ and as automated patching solutions become more cost-effective to users, patching rights should be taxed relatively more heavily. Further,*

(i) if $\frac{(\pi_a \alpha_a)^2}{1-\pi_a \alpha_a} < c_a \le c_p - \pi_a \alpha_a$, then optimally taxed patching rights decrease the size of the unpatched population by $\frac{1}{2\pi_s \alpha_s} + K_l$ such that $SL_P < SL_{SQ}$, decrease the size of the automated patching population by $\frac{1}{4(1-\pi_a \alpha_a)\pi_s \alpha_s} + K_m$ such that $AL_P < AL_{SQ}$, and increase social welfare by $\frac{c_a}{4(1-\pi_a \alpha_a)\pi_s \alpha_s} + K_n$;

(ii) if $c_a \le \min\left(c_p - \pi_a \alpha_a, \frac{(\pi_a \alpha_a)^2}{1-\pi_a \alpha_a}\right)$, then optimally taxed patching rights decrease the size of the unpatched population by $\frac{2-c_a-\pi_a \alpha_a}{2\pi_s \alpha_s} + K_o$ such that $SL_P < SL_{SQ}$ if and only if $c_a > \frac{2\left(1-\sqrt{1-\pi_a \alpha_a (1-\pi_a \alpha_a)}\right)-\pi_a \alpha_a (1-\pi_a \alpha_a)}{1-\pi_a \alpha_a}$, decrease the size of the automated patching population by $\frac{c_a+\pi_a \alpha_a}{2\pi_s \alpha_s} + K_p$ such that $AL_P < AL_{SQ}$, and increase social welfare by $\frac{(c_a+\pi_a \alpha_a)^2}{4\pi_s \alpha_s} + K_q$.

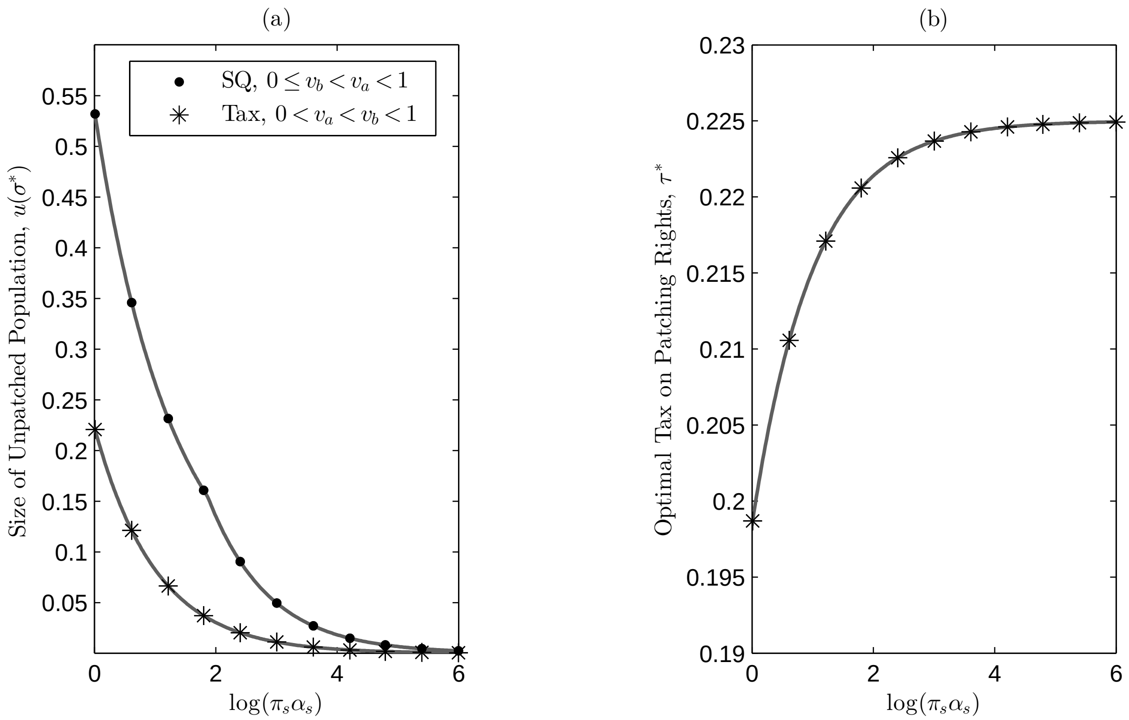

Figure 4: How the optimal tax on patching rights and associated unpatched usage in equilibrium are influenced by the effective security loss factor. Panel (a) illustrates the size of the unpatched population under both the status quo and in the presence of the tax. The optimal tax is depicted in panel (b). The parameter values are $\alpha_a = 3.5, \pi_a = 0.1, c_p = 0.5,$ and $c_a = 0.1$.

As Proposition 7 indicates, the socially optimal tax in this case is structured with a different concern in mind. In particular, Lemma 4 establishes that when c_a becomes lower, the equilibrium consumer market structure under the status quo is characterized by $0 < v_b < v_a < 1$ which is to say that consumers no longer find standard patching necessary due to the improvement in automated patching technology. In this case, parts (ii) and (iii) of Lemma 5 demonstrate that an optimally configured tax on patching rights will either induce the same consumer market structure or even $0 < v_a < v_b < 1$ if c_a is sufficiently reduced. In neither case can the planner induce standard patching behavior despite the effective security loss factor potentially being high. Therefore, in contrast to our findings in Proposition 6, the role of the optimal tax here is fundamentally different. Because high valuation users can have natural incentives to choose automated patching and thus need not be compelled to retain patching rights, the social planner has the ability to leverage high taxes on patching rights without affecting their behavior. For example, these consumers will choose not to retain patching rights under the conditions of part (i) of Proposition 7, in that $0 < v_b < v_a < 1$ is induced in equilibrium. As the proposition states, this large tax helps to significantly reduce unpatched usage by lower valuation consumers and increase social welfare. Because a large mass of lower valuation users are pushed out of the market by the tax, some users of the automated patching option under the status quo now switch to being unpatched, hence the size of the automated patching population also shrinks relative to the status quo.

More generally, we find that as automated patching solutions improve (across regions), a social planner should tend to utilize larger taxes to disincentivize the retainment of patching rights and significantly throttle unpatched usage. Interestingly, part (ii) of Proposition 7 establishes that if this technology improves sufficiently, large taxes will actually result in higher valuation consumers strategically retaining patching rights (despite the premium) and remaining unpatched. In this case, they benefit from the fact that only a small mass of consumers will find this behavior in their best interest which, in turn, limits the expected security losses they will incur. In this case, as can be seen in part (iii) of Lemma 5, the optimal tax is larger than c_a. Despite this outcome being an improvement to social welfare, because higher valuation users are the ones incentivized to remain unpatched, expected security losses can be increased relative to the status quo.

5 Concluding Remarks

In the current state of affairs, both software end users and system administrators are faced with a barrage of security patches arriving weekly. However, because users are endowed with the right to choose whether or not to apply these security updates, a large portion of the user base ultimately chooses to remain unpatched, leaving their systems prone to security attacks. These users contribute to a security externality that affects all users of the software which degrades its value and has a negative impact on its profitability. In this paper, we propose an improved business model where a software vendor differentiates its product based on patching rights. In this model, the right to choose whether or not to patch is no longer endowed. Instead, consumers who prefer to retain these rights and hence control of the patching status of their systems must pay a relative premium. Consumers who prefer to relinquish these rights have their systems automatically updated by the vendor and, in exchange, pay a discounted price. The market segmentation induced carries a reduction in security risk and an increase in profitability to the vendor. In this way, the pricing of patching rights can be a beneficial marketing strategy driving revenue growth; a vendor can market its product offerings as being more secure because its differentiated products incentivize better security behaviors by users.

We find that a priced patching rights strategy is a very effective way to improve software security. We characterize the optimal manner in which to price these rights and demonstrate that it leads to a significant increase in the profits of firms producing proprietary software. Moreover, the size of unpatched users decreases in equilibrium under priced patching rights which, in turn, tends to lead to a decrease in losses associated with security attacks on systems running the software. In some cases, the firm may raise both the price of the premium version with patching rights and the price of the "discounted" version without patching rights relative to the optimal price offered under the status quo. This demonstrates how a software firm can extract value from a combination of improved automated patching technology and a pricing strategy that incentivizes better security outcomes that are valued by consumers. We establish that the pricing of patching rights can negatively affect social welfare when usage in the market significantly contracts, but in many cases welfare improves as a result of the lower resulting security risk.

Unlike with proprietary software, merely the inclusion of an automated patching option with open-source software often does not reduce the size of the unpatched population. In fact, under a high effective security loss factor, the size of the unpatched population is unchanged because consumers with lower valuations enter the market, re-establishing the externality. However, its inclusion does lead to a greater prevalence of automated patching as an equilibrium strategy across software classes (comparing OSS to proprietary software). Because of the large unpatched population that remains despite having an automated patching option, there is arguably an even greater need for patching rights to be managed in the OSS domain. We find that a tax on patching rights can be quite effective, and we characterize how it should be structured. In particular, when automated patching costs are comparable to standard patching costs and all segments are represented under the status quo, the optimal tax tends to be higher under a moderate effective security loss factor and more modest under a higher one. In this case, the tax is effective at reducing unpatched populations, expanding automated patching and achieving reduced security losses. When automated patching technology improves and costs become even smaller, a relatively large tax is warranted and leads to both a reduced unpatched population and a reduced automated patching population, in comparison to the status quo. Because of the significant drop in unpatched usage, in this case it is possible for expected security losses to increase when higher valuation consumers are the ones who become incentivized to remain unpatched in equilibrium. Nevertheless, the taxing of patching rights leads to a substantial improvement in social welfare.

We employ asymptotic analysis which is commonly employed in microeconomic studies. Its use can be expected here due to the complexity of the game and corresponding equilibrium characterization (some examples of studies using asymptotic analysis include Li et al. 1987, Laffont and Tirole 1988, MacLeod and Malcomson 1993, Pesendorfer and Swinkels 2000, Muller 2000, Tunca and Zenios 2006, August and Tunca 2006, Pei et al. 2011 among many others). Miller (2006) and Cormen et al. (2009) provide comprehensive treatments of the mathematical foundation underlying asymptotic analysis. In this study, we have characterized regions on which the results and implications related to our primary research questions arise. Because of model complexity in this setting, some boundaries of these regions do not have explicit functional forms. These boundaries would normally be implicit and would be characterized accordingly. However, the objective of the analysis is the identification of regions of applicability in terms of parameter characteristics, which is the focus of our formalized results.

Under priced patching rights, the consumers who pay a premium to retain these rights need no additional monitoring by the proprietary vendor or OSS provider to examine patching status. However, consumers who benefit financially in exchange for these rights must have their systems automatically updated per the contract. This requires a careful implementation that entails the monitoring of systems. First, the updating of systems need not be in full control by the vendor nor instantaneous to derive the benefits of this policy. For example, consumers can be given a 24 hour window to apply patches before they are forcefully installed. This gives users some leeway operationally. Second, a user can always disconnect her system from the network to avoid the deployment of automated security updates. In this case, although the patches are not installed, the externality imposed by this unpatched system is also removed because the system is no longer connected to the network.

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Online Supplement for
"Market Segmentation and Software Security: Pricing Patching Rights"

Appendix

1 Proprietary Software: Status Quo

Lemma A.1 *Under the status quo, i.e., $\delta = 1$, the complete threshold characterization of the consumer market equilibrium is as follows:*

(I) $(0 < v_b < 1)$, *where* $v_b = \frac{1}{2} + \frac{-1+\sqrt{(1-\pi_s\alpha_s)^2+4p\pi_s\alpha_s}}{2\pi_s\alpha_s}$*:*

- *(A)* $2c_a + 2\pi_a\alpha_a + \sqrt{(1-\pi_s\alpha_s)^2 + 4p\pi_s\alpha_s} \geq 1 + \pi_s\alpha_s$
- *(B)* $(1-\pi_a\alpha_a)(c_a + p\pi_a\alpha_a) - \pi_s\alpha_s(c_a + p)(1 - c_a - p - \pi_a\alpha_a) \geq 0$
- *(C)* *Either* $1 + \pi_s\alpha_s \leq 2c_p$, *or*
- *(D)* $1 + \pi_s\alpha_s > 2c_p$ *and* $c_p - (1 - c_p - p)\pi_s\alpha_s \geq c_p^2$

(II) $(0 < v_b < v_a < 1)$, *where* v_a *is the most positive root of the cubic* $f_4(x) \triangleq c_a^2 + c_a(-1 + 2\pi_a\alpha_a)x + x^3\pi_s\alpha_s(1-\pi_a\alpha_a) - x^2(\pi_a\alpha_a(1-\pi_a\alpha_a) + \pi_s\alpha_s(c_a+p))$ *and* $v_b = \frac{pv_a}{-c_a+v_a(1-\pi_a\alpha_a)}$*:*

- *(A)* $c_a^2 - (1-\pi_a\alpha_a)(\pi_a\alpha_a - \pi_s\alpha_s) - c_a(1 - 2\pi_a\alpha_a + \pi_s\alpha_s) > \pi_s\alpha_s p$
- *(B)* $\pi_a\alpha_a \leq c_p - c_a$

(III) $(0 < v_b < v_a < v_p < 1)$, *where* v_a *is the most positive root of* $f_4(x)$ *and* $v_b = \frac{pv_a}{-c_a+v_a(1-\pi_a\alpha_a)}$ *and* $v_p = \frac{c_p - c_a}{\pi_a\alpha_a}$*:*

- *(A)* $c_p < c_a + \pi_a\alpha_a$
- *(B)* $c_p(1-\pi_a\alpha_a) > c_a$
- *(C)* $c_p(\pi_a\alpha_a)^2(-c_a + c_p(1-\pi_a\alpha_a)) + \pi_s\alpha_s(-c_a + c_p)^2(c_a - c_p(1-\pi_a\alpha_a) + \pi_a\alpha_a p) < 0$
- *(D)* $\pi_s\alpha_s(c_a - c_p)^2(c_a - c_p(1-\pi_a\alpha_a) + \pi_a\alpha_a p) < c_p(\pi_a\alpha_a)^2(c_a - c_p(1-\pi_a\alpha_a))$

(IV) $(0 < v_b < v_p < 1)$, *where* v_b *is the most positive root of* $f_3(x)$ *and* $v_p = \frac{c_p v_b}{v_b - p}$*:*

- *(A)* $c_p^2 > c_p - (1 - c_p - p)\pi_s\alpha_s$
- *(B)* $(1-\pi_a\alpha_a)(c_a + \pi_a\alpha_a p)^2 + \pi_s\alpha_s(c_a+p)^2(c_a - c_p(1-\pi_a\alpha_a) + \pi_a\alpha_a p) \geq 0$
- *(C)* *Either* $c_a - c_p(1-\pi_a\alpha_a) \geq 0$, *or*
- *(D)* $c_a - c_p(1-\pi_a\alpha_a) < 0$ *and* $c_p(1-\pi_a\alpha_a) > c_a$ *and* $c_p(\pi_a\alpha_a)^2(-c_a + c_p(1-\pi_a\alpha_a)) + \pi_s\alpha_s(-c_a + c_p)^2(c_a - c_p(1-\pi_a\alpha_a) + \pi_a\alpha_a p) \geq 0$

Proof of Lemma A.1: This is proven as a sub-case in the proof of Lemma A.2. $\square$

Proof of Lemma 2: Technically, we prove the existence of $\tilde{\alpha}_1$ such that for $\alpha_s > \tilde{\alpha}_1$, p^* is set so that

1. if $c_p - \pi_a\alpha_a < c_a < 1 - \pi_a\alpha_a - (1-c_p)\sqrt{1-\pi_a\alpha_a}$, then $\sigma^*(v)$ is characterized by $0 < v_b < v_a < v_p < 1$ under optimal pricing, and
2. if $c_a > 1 - \pi_a\alpha_a - (1-c_p)\sqrt{1-\pi_a\alpha_a}$, then $\sigma^*(v)$ is characterized by $0 < v_b < v_p < 1$ under optimal pricing.

Because $c_p - \pi_a\alpha_a < c_a$, part (II) of Lemma A.1 cannot be satisfed. We consider the three, remaining possibilities. Suppose $0 < v_b < 1$. By part (I) of Lemma A.1, we obtain $v_b = 1 - \frac{1-p}{\pi_s\alpha_s} + \frac{(1-p)p}{(\pi_s\alpha_s)^2} + O\left(\frac{1}{(\pi_s\alpha_s)^3}\right)$ which upon substitution into (6) yields

$$\Pi(p) = \frac{1-2p}{\pi_s\alpha_s} - \frac{p(2-3p)}{(\pi_s\alpha_s)^2} - \frac{p(2-(9-8p)p)}{(\pi_s\alpha_s)^3} + O\left(\frac{1}{(\pi_s\alpha_s)^4}\right). \tag{A.1}$$

Its unconstrained maximizer is given by

$$p_N = \frac{1}{2} - \frac{1}{8\pi_s\alpha_s} + \frac{1}{16(\pi_s\alpha_s)^2} + O\left(\frac{1}{(\pi_s\alpha_s)^3}\right). \tag{A.2}$$

Substituting (A.2) into (A.1), we obtain

$$\Pi_N = \frac{1}{4\pi_s\alpha_s} - \frac{1}{8(\pi_s\alpha_s)^2} + O\left(\frac{1}{(\pi_s\alpha_s)^3}\right). \tag{A.3}$$

Suppose $0 < v_b < v_a < v_p < 1$ is induced. By part (III) of Lemma A.1, we obtain $v_b = \frac{c_a+p}{1-\pi_a\alpha_a} - \frac{c_a(c_a+p\pi_a\alpha_a)}{(c_a+p)^2\pi_s\alpha_s} + \frac{c_a p(1-\pi_a\alpha_a)(2c_a-(c_a-p)\pi_a\alpha_a)(c_a+p\pi_a\alpha_a)}{(c_a+p)^5(\pi_s\alpha_s)^2} + O\left(\frac{1}{(\pi_s\alpha_s)^3}\right)$, which upon substitution into (6) yields

$$\begin{aligned}\Pi(p) = {} & \frac{p(1-\pi_a\alpha_a-c_a-p)}{1-\pi_a\alpha_a} + \frac{c_a p(c_a+p\pi_a\alpha_a)}{(c_a+p)^2\pi_s\alpha_s} - \\ & \frac{c_a p^2(1-\pi_a\alpha_a)(2c_a-(c_a-p)\pi_a\alpha_a)(c_a+p\pi_a\alpha_a)}{(c_a+p)^5(\pi_s\alpha_s)^2} + O\left(\frac{1}{(\pi_s\alpha_s)^3}\right).\end{aligned} \tag{A.4}$$

Its unconstrained maximizer is given by

$$p_A = \frac{1}{2}(1-\pi_a\alpha_a-c_a) + \frac{4(c_a^2(1-\pi_a\alpha_a)(c_a-\frac{1}{2}(1-\pi_a\alpha_a-c_a)(1-2\pi_a\alpha_a)))}{(1-\pi_a\alpha_a+c_a)^3\pi_s\alpha_s} + O\left(\frac{1}{(\pi_s\alpha_s)^2}\right). \tag{A.5}$$

Substituting (A.5) into (A.4), we obtain

$$\Pi_A = \frac{(1-\pi_a\alpha_a-c_a)^2}{4(1-\pi_a\alpha_a)} + \frac{c_a(1-\pi_a\alpha_a-c_a)(c_a(2-\pi_a\alpha_a)+\pi_a\alpha_a(1-\pi_a\alpha_a))}{(1-\pi_a\alpha_a+c_a)^2\pi_s\alpha_s} + O\left(\frac{1}{(\pi_s\alpha_s)^2}\right). \tag{A.6}$$

Lastly, suppose $0 < v_b < v_p < 1$ is induced. By part (IV) of Lemma A.1, we obtain $v_b = c_p + p - \frac{c_p^2}{(c_p+p)^2\pi_s\alpha_s} + \frac{2c_p^3 p}{(c_p+p)^5(\pi_s\alpha_s)^2} + O\left(\frac{1}{(\pi_s\alpha_s)^3}\right)$, which upon substitution into (6) yields

$$\Pi(p) = p(1 - c_p - p) + \frac{c_p^2 p}{(c_p + p)^2\pi_s\alpha_s} - \frac{2c_p^3 p^2}{(c_p + p)^5(\pi_s\alpha_s)^2} + O\left(\frac{1}{(\pi_s\alpha_s)^3}\right). \tag{A.7}$$

Its unconstrained maximizer is given by

$$p_P = \frac{1 - c_p}{2} - \frac{2c_p^2(1 - 3c_p)}{(1 + c_p)^3\pi_s\alpha_s} - \frac{16c_p^3(-3 + 8c_p - 5c_p^2 + 8c_p^3)}{(1 + c_p)^7(\pi_s\alpha_s)^2} + O\left(\frac{1}{(\pi_s\alpha_s)^3}\right). \tag{A.8}$$

Substituting (A.8) into (A.7), we obtain

$$\Pi_P = \frac{1}{4}(1 - c_p)^2 + \frac{2(1 - c_p)c_p^2}{(1 + c_p)^2\pi_s\alpha_s} + O\left(\frac{1}{(\pi_s\alpha_s)^2}\right). \tag{A.9}$$

Comparing (A.3) with either (A.6) or (A.9), we find that (A.3) is dominated by the other two profits when α_s is sufficiently large (say, $\alpha_s > \hat{\alpha}_1$, for some $\hat{\alpha}_1$). Then using (A.5) with Lemma A.1, we find that, for sufficiently large α_s ($\alpha_s > \hat{\alpha}_2$, for some $\hat{\alpha}_2$), the optimal price of this case indeed induces the correct market structure when $c_p - \pi_a\alpha_a < c_a < \frac{(2c_p-\pi_a\alpha_a)(1-\pi_a\alpha_a)}{2-\pi_a\alpha_a}$. Similarly, using (A.8) with Lemma A.1, we find that, for sufficiently large α_s ($\alpha_s > \hat{\alpha}_3$, for some $\hat{\alpha}_3$), the optimal price of this case indeed induces the correct market structure when $c_a > c_p - \frac{1}{2}(1 + c_p)\pi_a\alpha_a$. Let $\tilde{\alpha}_1$ be the max of these $\hat{\alpha}_1$, $\hat{\alpha}_2$, and $\hat{\alpha}_3$. Then for $\alpha_s > \tilde{\alpha}_1$, if $c_a > c_p - \pi_a\alpha_a$, we have that (A.3) is dominated by the other two profits, at least one of which has its profit-maximizing solution inducing the market structure it represents. Moreover, since $c_p - \frac{1}{2}(1 + c_p)\pi_a\alpha_a < \frac{(2c_p-\pi_a\alpha_a)(1-\pi_a\alpha_a)}{2-\pi_a\alpha_a}$, there will be a region in which both interior maximizers induce their corresponding cases.

Comparing (A.6) with (A.9), we see that (A.6) has greater profit when $2c_a(1-\pi_a\alpha_a) < c_a^2+(1-\pi_a\alpha_a)(c_p(2-c_p)-\pi_a\alpha_a)$, which can be written as $c_a < 1-\pi_a\alpha_a-(1-c_p)\sqrt{1 - \pi_a\alpha_a}$. Note that for any $c_p \in [0, 1]$, we have that $c_p-\frac{1}{2}(1+c_p)\pi_a\alpha_a < 1-\pi_a\alpha_a-(1-c_p)\sqrt{1 - \pi_a\alpha_a} < \frac{(2c_p-\pi_a\alpha_a)(1-\pi_a\alpha_a)}{2-\pi_a\alpha_a}$ from $\pi_a\alpha_a \in (0, 1)$. Therefore, for $\alpha_s > \tilde{\alpha}_1$, if $c_p-\pi_a\alpha_a < c_a \le 1-\pi_a\alpha_a-(1-c_p)\sqrt{1 - \pi_a\alpha_a}$, then $\sigma^*(v)$ is characterized by $0 < v_b < v_a < v_a < 1$ under optimal pricing, and if $c_a > 1-\pi_a\alpha_a-(1-c_p)\sqrt{1 - \pi_a\alpha_a}$, then $\sigma^*(v)$ is characterized by $0 < v_b < v_p < 1$ under optimal pricing. $\square$

2 Proprietary Software: Patching Rights Priced

Lemma A.2 *Under priced patching rights, the complete threshold characterization of the consumer market equilibrium is as follows:*

(I) $(0 < v_a < 1)$, *where* $v_a = \frac{\delta p + c_a}{1-\pi_a\alpha_a}$:

- *(A)* $(1-\delta)p \geq \pi_a\alpha_a + c_a$

(II) $(0 < v_a < v_b < 1)$, *where* $v_a = \frac{\delta p + c_a}{1-\pi_a\alpha_a}$ *and* $v_b = \frac{1}{2} + \frac{-\pi_a\alpha_a + \sqrt{(\pi_a\alpha_a - \pi_s\alpha_s)^2 + 4\pi_s\alpha_s((1-\delta)p - c_a)}}{2\pi_s\alpha_s}$:

- *(A)* $\pi_a\alpha_a + c_a > (1-\delta)p > c_a$
- *(B)* $(2\pi_s\alpha_s)\left(\frac{c_a+\delta p}{1-\pi_a\alpha_a}\right) + \pi_a\alpha_a - \pi_s\alpha_s < \sqrt{(\pi_a\alpha_a - \pi_s\alpha_s)^2 + 4\pi_s\alpha_s((1-\delta)p - c_a)}$
- *(C) Either* $\pi_a\alpha_a + \pi_s\alpha_s \leq 2c_p$, *or*
- *(D)* $\pi_a\alpha_a + \pi_s\alpha_s > 2c_p$ *and* $(1-\delta)p \geq c_a + \frac{(c_p - \pi_a\alpha_a)(c_p - \pi_s\alpha_s)}{\pi_s\alpha_s}$

(III) $(0 < v_a < v_b < v_p < 1)$, *where* $v_a = \frac{\delta p + c_a}{1-\pi_a\alpha_a}$ *and* v_b *is the most positive root of* $f_1(x) \triangleq (c_a - (1-\delta)p + x\pi_a\alpha_a)^2 + x^2\pi_s\alpha_s(c_a - c_p - (1-\delta)p + x\pi_a\alpha_a)$, *and* $v_p = \frac{c_p v_b}{c_a - (1-\delta)p + v_b\pi_a\alpha_a}$:

- *(A)* $c_a + \frac{(c_p - \pi_a\alpha_a)(c_p - \pi_s\alpha_s)}{\pi_s\alpha_s} > (1-\delta)p > c_a$
- *(B)* $\pi_a\alpha_a > c_p$
- *(C) Either* $(1-\delta-\pi_a\alpha_a)p \geq c_a$, *or*
- *(D)* $(1-\delta-\pi_a\alpha_a)p < c_a$, *and* $\pi_s\alpha_s(c_a+\delta p)^2(c_a - c_p(1-\pi_a\alpha_a) - (1-\delta-\pi_a\alpha_a)p) + (1-\pi_a\alpha_a)(c_a - (1-\delta-\pi_a\alpha_a)p)^2 < 0$

(IV) $(0 < v_b < 1)$, *where* $v_b = \frac{1}{2} + \frac{-1+\sqrt{(1-\pi_s\alpha_s)^2 + 4p\pi_s\alpha_s}}{2\pi_s\alpha_s}$:

- *(A)* $2c_a - 2(1-\delta)p + 2\pi_a\alpha_a + \sqrt{(1-\pi_s\alpha_s)^2 + 4\pi_s\alpha_s p} \geq 1 + \pi_s\alpha_s$
- *(B)* $(1-\pi_a\alpha_a)(c_a - (1-\delta-\pi_a\alpha_a)p) - \pi_s\alpha_s(c_a+\delta p)(1 - c_a - \delta p - \pi_a\alpha_a) \geq 0$
- *(C) Either* $1 + \pi_s\alpha_s \leq 2c_p$, *or*
- *(D)* $1 + \pi_s\alpha_s > 2c_p$ *and* $c_p - (1 - c_p - p)\pi_s\alpha_s \geq c_p^2$

(V) $(0 < v_b < v_a < 1)$, *where* v_a *is the most positive root of the cubic* $f_2(x) \triangleq c_a^2 - 2c_a(1-\delta)p + ((1-\delta)p)^2 + x(c_a - (1-\delta)p)(-1+2\pi_a\alpha_a) + x^3\pi_s\alpha_s(1-\pi_a\alpha_a) - x^2(\pi_a\alpha_a(1-\pi_a\alpha_a) + \pi_s\alpha_s(c_a + p\delta))$ *and* $v_b = \frac{pv_a}{-c_a + (1-\delta)p + v_a(1-\pi_a\alpha_a)}$:

- *(A)* $(1-\delta)p < c_a$
- *(B)* $c_a^2 + ((1-\delta)p)^2 + (1-\delta)p(1-2\pi_a\alpha_a) - (1-\pi_a\alpha_a)(\pi_a\alpha_a - \pi_s\alpha_s) - c_a(1 + 2(1-\delta)p - 2\pi_a\alpha_a + \pi_s\alpha_s) > \pi_s\alpha_s\delta p$
- *(C)* $\pi_a\alpha_a \leq (1-\delta)p + c_p - c_a$

(VI) $(0 < v_b < v_a < v_p < 1)$, *where* v_a *is the most positive root of* $f_2(x)$ *and* $v_b = \frac{pv_a}{-c_a + (1-\delta)p + v_a(1-\pi_a\alpha_a)}$ *and* $v_p = \frac{c_p - (c_a - (1-\delta)p)}{\pi_a\alpha_a}$:

(A) $(1-\delta)p < c_a$

(B) $c_p + (1-\delta)p < c_a + \pi_a\alpha_a$

(C) $c_p(1-\pi_a\alpha_a) + p(1-\delta) > c_a$

(D) $c_p(\pi_a\alpha_a)^2(-c_a + c_p(1-\pi_a\alpha_a) + p(1-\delta)) + \pi_s\alpha_s(-c_a + c_p + p(1-\delta))^2(c_a - c_p(1-\pi_a\alpha_a) - (1-\delta-\pi_a\alpha_a)p) < 0$

(VII) $(0 < v_b < v_p < 1)$, *where* v_b *is the most positive root of the cubic* $f_3(x) \triangleq p^2 + x^2(1 - c_p\pi_s\alpha_s + \pi_s\alpha_s x) - px(2 + \pi_s\alpha_s x)$ *and* $v_p = \frac{c_p v_b}{v_b - p}$*:*

(A) $c_a > (1-\delta-\pi_a\alpha_a)p$

(B) $c_p^2 > c_p - (1-c_p-p)\pi_s\alpha_s$

(C) $(1-\pi_a\alpha_a)(c_a-(1-\delta-\pi_a\alpha_a)p)^2 + \pi_s\alpha_s(c_a+\delta p)^2(c_a - c_p(1-\pi_a\alpha_a) - (1-\delta-\pi_a\alpha_a)p) \geq 0$

(D) *Either* $c_a - (1-\delta)p - c_p(1-\pi_a\alpha_a) \geq 0$*, or*

(E) $c_a - (1-\delta)p - c_p(1-\pi_a\alpha_a) < 0$ *and* $c_p(1-\pi_a\alpha_a) + (1-\delta)p > c_a$*, and* $c_p(\pi_a\alpha_a)^2(-c_a + c_p(1-\pi_a\alpha_a) + p(1-\delta)) + \pi_s\alpha_s(-c_a + c_p + p(1-\delta))^2(c_a - c_p(1-\pi_a\alpha_a) - (1-\delta-\pi_a\alpha_a)p) \geq 0$

Proof of Lemma A.2: First, we establish the general threshold-type equilibrium structure. Given the size of unpatched user population u, the net payoff of the consumer with type v for strategy profile σ is written as

$$U(v,\sigma) \triangleq \begin{cases} v - p - c_p & if \quad \sigma(v) = (B,P)\,; \\ v - p - \pi_s\alpha_s uv & if \quad \sigma(v) = (B,NP)\,; \\ v - \delta p - c_a - \pi_a\alpha_a v & if \quad \sigma(v) = (B,AP)\,; \\ 0 & if \quad \sigma(v) = (NB,NP)\,. \end{cases} \tag{A.10}$$

Note $\sigma(v) = (B,P)$ if and only if

$$v - p - c_p \geq v - p - \pi_s\alpha_s uv \Leftrightarrow v \geq \frac{c_p}{\pi_s\alpha_s u}, \text{ and}$$

$$v - p - c_p \geq v - \delta p - c_a - \pi_a\alpha_a v \Leftrightarrow v \geq \frac{(1-\delta)p + c_p - c_a}{\pi_a\alpha_a}, \text{ and}$$

$$v - p - c_p \geq 0 \Leftrightarrow v \geq c_p + p,$$

which can be summarized as

$$v \geq \max\left(\frac{c_p}{\pi_s\alpha_s u}, \frac{(1-\delta)p + c_p - c_a}{\pi_a\alpha_a}, c_p + p\right). \tag{A.11}$$

By (A.11), if a consumer with valuation v_0 buys and patches the software, then every consumer with valuation $v > v_0$ will also do so. Hence, there exists a threshold $v_p \in (0,1]$ such that for all $v \in \mathcal{V}$, $\sigma^*(v) = (B,P)$ if and only if $v \geq v_p$. Similarly, $\sigma(v) \in \{(B,P),(B,NP),(B,AP)\}$, i.e., the consumer purchases one of the alternatives, if and only if

$$v - p - c_p \geq 0 \Leftrightarrow v \geq c_p + p, \text{ or}$$

$$v - p - \pi_s \alpha_s u v \geq 0 \Leftrightarrow v \geq \frac{p}{1 - \pi_s \alpha_s u}, \text{ or}$$

$$v - \delta p - c_a - \pi_a \alpha_a v \geq 0 \Leftrightarrow v \geq \frac{\delta p + c_a}{1 - \pi_a \alpha_a},$$

which can be summarized as

$$v \geq \min\left(c_p + p, \frac{p}{1 - \pi_s \alpha_s u}, \frac{\delta p + c_a}{1 - \pi_a \alpha_a}\right). \tag{A.12}$$

Let $0 < v_1 \leq 1$ and $\sigma^*(v) \in \{(B, P), (B, NP), (B, AP)\}$, then by (A.12), for all $v > v_1$, $\sigma^*(v) \in \{(B, P), (B, NP), (B, AP)\}$, and hence there exists a $\underline{v} \in (0, 1]$ such that a consumer with valuation $v \in \mathcal{V}$ will purchase if and only if $v \geq \underline{v}$.

By (A.11) and (A.12), $\underline{v} \leq v_p$ holds. Moreover, if $\underline{v} < v_p$, consumers with types in $[\underline{v}, v_p]$ choose either (B, NP) or (B, AP). A purchasing consumer with valuation v will prefer (B, NP) over (B, AP) if and only if

$$v - p - \pi_s \alpha_s u v > v - \delta p - c_a - \pi_a \alpha_a v \Leftrightarrow v\left(\pi_a \alpha_a - \pi_s \alpha_s u\right) > (1 - \delta)p - c_a. \tag{A.13}$$

This inequality can be either $v > \frac{(1-\delta)p - c_a}{\pi_a \alpha_a - \pi_s \alpha_s u}$ or $v < \frac{(1-\delta)p - c_a}{\pi_a \alpha_a - \pi_s \alpha_s u}$, depending on the sign of $\pi_a \alpha_a - \pi_s \alpha_s u$. Consequently, there can be two cases for (B, NP) and (B, AP) in equilibrium: first, there exists $v_u \in [\underline{v}, v_p]$ such that $\sigma(v) = (B, NP)$ for all $v \in [v_u, v_p)$, and $\sigma(v) = (B, AP)$ for all $v \in [v_d, v_u)$ where $v_d = \underline{v}$. In the second case, there exists $v_d \in [\underline{v}, v_p]$ such that $\sigma(v) = (B, AP)$ for all $v \in [v_d, v_p)$, and $\sigma(v) = (B, NP)$ for all $v \in [v_u, v_d)$, where $v_u = \underline{v}$. If $\pi_a \alpha_a - \pi_s \alpha_s u = 0$, then depending on the sign of $(1 - \delta)p - c_a$, all consumers unilaterally prefer either (B, NP) or (B, AP); e.g., if $(1-\delta)p > c_a$, all consumers prefer (B, AP), and if $(1-\delta)p < c_a$, then all consumers prefer (B, NP). Finally, if $(1 - \delta)p = c_a$, then all consumers are indifferent between (B, NP) and (B, AP), in which case only the size of the consumer population u matters in equilibrium, i.e., $\pi_a \alpha_a - \pi_s \alpha_s u = 0$ in equilibrium. Technically, there are multiple equilibria in this case; however, utility of each consumer and the vendor profit are the same in all equilibria. So, without loss of generality, we focus on the threshold-type equilibrium in this case. In summary, we have established the threshold-type consumer market equilibrium structure.

Next, we characterize in more detail each outcome that can arise in equilibrium, as well as the corresponding parameter regions. For Case (I), in which all consumers who purchase choose the automated patching option, i.e., $0 < v_a < 1$, based on the threshold-type equilibrium structure, we have $u = 0$. We prove the following claim related to the corresponding parameter region in which case (I) arises.

Claim 1 *The equilibrium that corresponds to case (I) arises if and only if the following condition is satisfied:*

$$(1 - \delta)p \geq \pi_a \alpha_a + c_a. \tag{A.14}$$

In this case, the threshold for the consumer indifferent between purchasing the automated patching option and not purchasing at all, v_a, satisfies

$$v_a = \frac{\delta p + c_a}{1 - \pi_a \alpha_a}. \tag{A.15}$$

For this to be an equilibrium, we need $0 < v_a < 1$, type $v = 1$ prefers (B, AP) over (B, P), and $v = 1$ prefers (B, AP) over (B, NP). Note that type $v = 1$ preferring (B, AP) over (B, P) implies that $v < 1$ does the same, by (A.11). Also, type $v = 1$ preferring (B, AP) over (B, NP) implies that all $v < 1$ does too. This is because $u = 0$, and so $\pi_a\alpha_a - \pi_s\alpha_s u > 0$, which implies that consumers of higher types prefer (B, NP) over (B, AP) by (A.13). Moreover, type $v = v_a$ preferring (NB, NP) over both (B, NP) and (B, P) implies that all $v < v_a$ do the same, by (A.12).

We always have $v_a > 0$ from our model assumptions, namely $\pi_a\alpha_a < 1$. To have $v_a < 1$, a necessary and sufficient condition is $\delta p + c_a + \pi_a\alpha_a < 1$.

For $v = 1$ to weakly prefer (B, AP) over (B, P), a necessary and sufficient condition is $1 - \delta p - c_a - \pi_a\alpha_a \geq 1 - p - c_p$, which reduces to $(1 - \delta)p \geq \pi_a\alpha_a + c_a - c_p$.

For $v = 1$ to weakly prefer (B, AP) over (B, NP), a necessary and sufficient condition is $1 - \delta p - c_a - \pi_a\alpha_a \geq 1 - p - \pi_s\alpha_s u$, which reduces down to $(1 - \delta)p \geq \pi_a\alpha_a + c_a$, using $u = 0$ in equilibrium.

Note that the condition $\delta p + c_a + \pi_a\alpha_a < 1$ is redundant, given $(1-\delta)p \geq \pi_a\alpha_a + c_a$ since the first condition can be written as $1 - \delta p > c_a + \pi_a\alpha_a$. Since $p < 1$, it follows that the $(1-\delta)p \geq \pi_a\alpha_a + c_a$ implies $1 - \delta p > c_a + \pi_a\alpha_a$. So case (I) arises if and only if the condition in (A.14) occurs. □

Next, for case (II), in which there are no consumers purchasing (B, P) but the upper tier of consumers are unpatched, i.e., $0 < v_a < v_b < 1$, we have $u = 1 - v_b$. Following the same steps as before, we prove the following claim related to the corresponding conditions for which case (II) arises.

Claim 2 *The equilibrium that corresponds to case (II) arises if and only if the following conditions are satisfied:*

$$\begin{gathered}\pi_a\alpha_a + c_a > (1-\delta)p > c_a \text{ and} \\ (2\pi_s\alpha_s)\left(\frac{c_a + \delta p}{1 - \pi_a\alpha_a}\right) + \pi_a\alpha_a - \pi_s\alpha_s < \sqrt{(\pi_a\alpha_a - \pi_s\alpha_s)^2 + 4\pi_s\alpha_s((1-\delta)p - c_a)} \text{ and} \\ \left\{(\pi_a\alpha_a + \pi_s\alpha_s \leq 2c_p) \text{ or } \left(\pi_a\alpha_a + \pi_s\alpha_s > 2c_p \text{ and } (1-\delta)p \geq c_a + \frac{(c_p - \pi_a\alpha_a)(c_p - \pi_s\alpha_s)}{\pi_s\alpha_s}\right)\right\}.\end{gathered} \tag{A.16}$$

In this case, the threshold for the consumer indifferent between purchasing the automated patching option and not purchasing at all, v_a, again satisfies

$$v_a = \frac{\delta p + c_a}{1 - \pi_a\alpha_a}. \tag{A.17}$$

The threshold for the consumer indifferent between being unpatched an purchasing the automated patching option, v_b, satisfies

$$v_b = \frac{(1-\delta)p - c_a}{\pi_a\alpha_a - \pi_s\alpha_s u}. \tag{A.18}$$

Using $u = 1 - v_b$, we find that v_b solves a quadratic equation. To find which root is the solution, we note that v_b must satisfy $\pi_a\alpha_a - \pi_s\alpha_s u > 0$ since higher types choose (B, NP) over (B, AP) by

(A.13). This implies $(1-\delta)p - c_a > 0$ in order for $v_b > 0$. Using this, we find that the root of the quadratic which specifies v_b is given by

$$v_b = \frac{\pi_s\alpha_s - \pi_a\alpha_a + \sqrt{(\pi_a\alpha_a - \pi_s\alpha_s)^2 + 4\pi_s\alpha_s((1-\delta)p - c_a)}}{2\pi_s\alpha_s}. \tag{A.19}$$

For this to be an equilibrium, we need $0 < v_a < v_b < 1$ and type $v = 1$ prefers (B, NP) over (B, P). Note that type $v = 1$ preferring (B, NP) over (B, P) implies that $v < 1$ does the same, by (A.11). This also implies that v_b prefers (B, AP) over (B, P), so that $v < v_b$ also prefer (B, AP) over (B, P), again by (A.11). Moreover, type $v = v_a$ preferring (NB, NP) over both (B, NP) and (B, P) implies that all $v < v_a$ do the same, by (A.12).

Again, we always have $v_a > 0$ from our model assumptions, namely $\pi_a\alpha_a < 1$. For $v_b < 1$, an equivalent condition is $(1-\delta)p < c_a + \pi_a\alpha_a$.

For $v_a < v_b$, it is equivalent to require $\sqrt{(\pi_a\alpha_a - \pi_s\alpha_s)^2 + 4\pi_s\alpha_s((1-\delta)p - c_a)} > (2\pi_s\alpha_s)\left(\frac{c_a+\delta p}{1-\pi_a\alpha_a}\right) + \pi_a\alpha_a - \pi_s\alpha_s$.

For type $v = 1$ to weakly prefer (B, NP) over (B, P), we equivalently have the condition $\pi_a\alpha_a + \pi_s\alpha_s - 2c_p \le \sqrt{(\pi_a\alpha_a - \pi_s\alpha_s)^2 + 4\pi_s\alpha_s((1-\delta)p - c_a)}$. This can be broken up into two cases, depending on whether $\pi_a\alpha_a + \pi_s\alpha_s - 2c_p \le 0$, and simplified in (A.16). $\square$

Next, for case (III), in which all segments are represented and the middle tier is unpatched, i.e., $0 < v_a < v_b < v_p < 1$, we have $u = v_p - v_b$. Following the same steps as before, we prove the following claim related to the corresponding parameter region in which case (III) arises.

Claim 3 *The equilibrium that corresponds to case (III) arises if and only if the following conditions are satisfied:*

$$c_a + \frac{(c_p - \pi_a\alpha_a)(c_p - \pi_s\alpha_s)}{\pi_s\alpha_s} > (1-\delta)p > c_a \text{ and } \pi_a\alpha_a > c_p \text{ and}$$
$$\Bigg\{\Big((1-\delta-\pi_a\alpha_a)p \ge c_a\Big) \text{ or } \Big((1-\delta-\pi_a\alpha_a)p < c_a \text{ and}$$
$$\pi_s\alpha_s(c_a+\delta p)^2(c_a - c_p(1-\pi_a\alpha_a) - (1-\delta-\pi_a\alpha_a)p) + (1-\pi_a\alpha_a)(c_a - (1-\delta-\pi_a\alpha_a)p)^2 < 0\Big)\Bigg\}. \tag{A.20}$$

In this case, the threshold for the consumer indifferent between purchasing the automated patching option and not purchasing at all, v_a, again satisfies

$$v_a = \frac{\delta p + c_a}{1 - \pi_a\alpha_a}. \tag{A.21}$$

To solve for the thresholds v_b and v_p, using $u = v_p - v_b$, note that they solve

$$v_b = \frac{(1-\delta)p - c_a}{\pi_a\alpha_a - \pi_s\alpha_s(v_p - v_b)}, \text{ and} \tag{A.22}$$

$$v_p = \frac{c_p}{\pi_s\alpha_s(v_p - v_b)}. \tag{A.23}$$

From (A.22), we have $\pi_s\alpha_s(v_p - v_b) = \pi_a\alpha_a - \frac{(1-\delta)p-c_a}{v_b}$, while from (A.23), we have $\pi_s\alpha_s(v_p - v_b) = \frac{c_p}{v_p}$. Equating these two expressions and solving for v_p in terms of v_b, we have

$$v_p = \frac{c_p v_b}{c_a - (1-\delta)p + v_b\pi_a\alpha_a}. \tag{A.24}$$

Plugging this expression for v_p into (A.23) and noting that $c_a - (1-\delta)p + v_b\pi_a\alpha_a > 0$ in order for $v_p > 0$, we find that v_b must be a zero of the cubic equation:

$$f_1(x) \triangleq (c_a - (1-\delta)p + x\pi_a\alpha_a)^2 + x^2\pi_s\alpha_s(c_a - c_p - (1-\delta)p + x\pi_a\alpha_a). \tag{A.25}$$

To find which root of the cubic v_b must be, first note that $\pi_a\alpha_a - \pi_s\alpha_s u > 0$ for consumers of higher valuation to prefer (B, NP) over (B, AP) by (A.13). From that, we have that $(1-\delta)p - c_a > 0$ in order for $v_b > 0$. To pin down the root of the cubic, note that the cubic's highest order term is $\pi_s\alpha_s\pi_a\alpha_a x^3$, so $\lim_{x\to-\infty} f_1(x) = -\infty$ and $\lim_{x\to\infty} f_1(x) = \infty$. We find $f_1(0) = ((1-\delta)p - c_a)^2 > 0$, $f_1\left(\frac{(1-\delta)p-c_a}{\pi_a\alpha_a}\right) = -\frac{c_p\pi_s\alpha_s((1-\delta)p-c_a)^2}{(\pi_a\alpha_a)^2} < 0$, and $f_1\left(\frac{(1-\delta)p-c_a+c_p}{\alpha_a}\right) = c_p^2 > 0$, while $0 < \frac{(1-\delta)p-c_a}{\pi_a\alpha_a} < \frac{(1-\delta)p-c_a+c_p}{\alpha_a}$.

We note that from (A.22), we have that $v_b > \frac{(1-\delta)p-c_a}{\pi_a\alpha_a}$, so it follows that v_b is the largest root of the cubic, lying between $\frac{(1-\delta)p-c_a}{\pi_a\alpha_a}$ and $\frac{(1-\delta)p-c_a+c_p}{\alpha_a}$. Then using (A.24), we solve for v_p.

For this to be an equilibrium, a necessary and sufficient condition is $0 < v_a < v_b < v_p < 1$. This tells us that all $v \in [v_p, 1]$ have the same preferences and will purchase (B, P), all $v \in [v_b, v_p)$ have the same preferences and will purchase (B, NP), and all $v \in [v_a, v_b)$ have the same preferences and will purchase (B, AP). Finally, all $v < v_a$ have the same preferences and will not purchase in equilibrium.

For $v_p < 1$, using (A.24), a necessary and sufficient condition for this to hold is $(1-\delta)p - c_a < v_b(\pi_a\alpha_a - c_p)$. Since $(1-\delta)p - c_a > 0$ (again, from $v_b > 0$), we need $\pi_a\alpha_a > c_p$. To have $v_b > \frac{(1-\delta)p-c_a}{\pi_a\alpha_a-c_p}$, a necessary and sufficient condition is that $f_1\left(\frac{(1-\delta)p-c_a}{\pi_a\alpha_a-c_p}\right) < 0$ so that the third root of $f_1(x)$ is greater than $\frac{(1-\delta)p-c_a}{\pi_a\alpha_a-c_p}$. Omitting the algebra, this simplifies to $c_a + \frac{(c_p-\pi_a\alpha_a)(c_p-\pi_s\alpha_s)}{\pi_s\alpha_s} > (1-\delta)p$.

For $v_b < v_p$, using (A.24), it is equivalent to have $v_b < \frac{(1-\delta)p-c_a+c_p}{\pi_a\alpha_a}$. A necessary and sufficient condition for this is that $f_1\left(\frac{(1-\delta)p-c_a+c_p}{\pi_a\alpha_a}\right) > 0$ so that the third root of $f_1(x)$ is smaller than $\frac{(1-\delta)p-c_a+c_p}{\pi_a\alpha_a}$. But $f_1\left(\frac{(1-\delta)p-c_a+c_p}{\pi_a\alpha_a}\right) = c_P^2 > 0$, so this always holds.

For $v_a < v_b$, using (A.15), an equivalent condition is $v_b > \frac{\delta p+c_a}{1-\pi_a\alpha_a}$. Since $v_b > \frac{(1-\delta)p-c_A}{\pi_a\alpha_a}$ (by the construction of v_b above as the largest root of the cubic), it follows that if $\frac{(1-\delta)p-c_A}{\pi_a\alpha_a} \geq \frac{\delta p+c_a}{1-\pi_a\alpha_a}$, then we don't need any extra conditions. The condition $\frac{(1-\delta)p-c_A}{\pi_a\alpha_a} \geq \frac{\delta p+c_a}{1-\pi_a\alpha_a}$ simplifies to $(1-\delta-\pi_a\alpha_a)p \geq c_a$. Otherwise, if $(1-\delta-\pi_a\alpha_a)p < c_a$, then we need $f_1\left(\frac{\delta p+c_a}{1-\pi_a\alpha_a}\right) < 0$ for $v_a < v_b$. This condition is $\pi_s\alpha_s(c_a+\delta p)^2(c_a - c_p(1-\pi_a\alpha_a) - (1-\delta-\pi_a\alpha_a)p) + (1-\pi_a\alpha_a)(c_a - (1-\delta-\pi_a\alpha_a)p)^2 < 0$, which is given in (A.20). $\square$

Next, for case (IV), in which all consumers who purchase are unpatched, i.e., $0 < v_b < 1$, we have $u = 1 - v_b$. Following the same steps as before, we prove the following claim related to the corresponding parameter conditions for which case (IV) arises.

Claim 4 *The equilibrium that corresponds to case (IV) arises if and only if the following conditions are satisfied:*

$$\begin{aligned}2c_a - 2(1-\delta)p + 2\pi_a\alpha_a + \sqrt{(1-\pi_s\alpha_s)^2 + 4\pi_s\alpha_s p} \geq 1 + \pi_s\alpha_s \ \textit{and}\\ (1-\pi_a\alpha_a)(c_a - (1-\delta-\pi_a\alpha_a)p) - \pi_s\alpha_s(c_a+\delta p)(1-c_a-\delta p-\pi_a\alpha_a) \geq 0 \ \textit{and}\\ \left\{\left(1+\pi_s\alpha_s \leq 2c_p\right) \ \textit{or} \ \left(1+\pi_s\alpha_s > 2c_p \ \textit{and} \ c_p - (1-c_p-p)\pi_s\alpha_s \geq c_p^2\right)\right\}.\end{aligned} \tag{A.26}$$

To solve for the threshold v_b, using $u = 1 - v_b$, we solve

$$v_b = \frac{p}{1-\pi_s\alpha_s(1-v_b}. \tag{A.27}$$

For this to be an equilibrium, we must have that $1 - \pi_s\alpha_s u > 0$, otherwise all consumers would prefer (NB, NP) over (B, NP), which can't happen in equilibrium. Using $1 - \pi_s\alpha_s(1 - v_b) > 0$, we find the right root of the quadratic for v_b to be

$$v_b = \frac{1}{2} + \frac{-1 + \sqrt{(1-\pi_s\alpha_s)^2 + 4p\pi_s\alpha_s}}{2\pi_s\alpha_s}. \tag{A.28}$$

For this outcome to be an equilibrium, the necessary and sufficient conditions are that $0 < v_b < 1$, type $v = 1$ prefers (B, NP) to both (B, AP) over (B, P), and $v = v_b$ to prefer (NB, NP) over (B, AP).

For $0 < v_b < 1$, it is equivalent to have $p < 1$, which is a model assumption.

For type $v = 1$ to weakly prefer (B, NP) to (B, AP), using the same steps as in the previous cases, the condition is $2c_a - 2(1-\delta)p + 2\pi_a\alpha_a + \sqrt{(1-\pi_s\alpha_s)^2 + 4\pi_s\alpha_s p} \geq 1 + \pi_s\alpha_s$.

For type $v = 1$ to weakly prefer (B, NP) to (B, P), omitting the algebra, the condition is $2c_p + \sqrt{(1-\pi_s\alpha_s)^2 + 4p\pi_s\alpha_s} \geq 1 + \pi_s\alpha_s$. This can be split into two cases, depending on the sign of $1 + \pi_s\alpha_s - 2c_p$ and given in (A.26).

Finally, for type $v = v_b$ to weakly prefer (NB, NP) to (B, AP), the condition is $\sqrt{(1-\pi_s\alpha_s)^2 + 4p\pi_s\alpha_s} \leq 1 + \frac{\pi_s\alpha_s(-1+2c_a+2p\delta+\pi_a\alpha_a)}{1-\pi_a\alpha_a}$. Since the right hand-side is always non-negative, $\pi_s\alpha_s > 0$, and $0 < \pi_a\alpha_a < 1$, this can be rewritten as $(1-\pi_a\alpha_a)(c_a - (1-\delta-\pi_a\alpha_a)p) - \pi_s\alpha_s(c_a+\delta p)(1-c_a-\delta p-\pi_a\alpha_a) \geq 0$. $\square$

Next, for case (V), in which the lower tier of purchasing consumers is unpatched while the upper tier does automated patching, i.e., $0 < v_b < v_a < 1$, we have $u = v_a - v_b$. Following the same steps as before, we prove the following claim related to the corresponding parameter region in which case (V) arises.

Claim 5 *The equilibrium that corresponds to case (V) arises if and only if the following conditions are satisfied:*

$$\begin{aligned}&(1-\delta)p < c_a \ \textit{and} \ \pi_a\alpha_a \leq (1-\delta)p + c_p - c_a \ \textit{and}\\ &c_a^2 + ((1-\delta)p)^2 + (1-\delta)p(1-2\pi_a\alpha_a) - (1-\pi_a\alpha_a)(\pi_a\alpha_a - \pi_s\alpha_s) - c_a(1+2(1-\delta)p - 2\pi_a\alpha_a + \pi_s\alpha_s) > \pi_s\alpha_s\delta p.\end{aligned} \tag{A.29}$$

To solve for the thresholds v_b and v_a, using $u = v_a - v_b$, note that they solve

$$v_b = \frac{p}{1 - \pi_s\alpha_s(v_a - v_b)}, \text{ and} \tag{A.30}$$

$$v_a = \frac{(1-\delta)p - c_a}{\pi_a\alpha_a - \pi_s\alpha_s(v_a - v_b)}. \tag{A.31}$$

From (A.30), we have $\pi_s\alpha_s(v_a - v_b) = 1 - \frac{p}{v_b}$, while from (A.31), we have $\pi_s\alpha_s(v_a - v_b) = \pi_a\alpha_a - \frac{(1-\delta)p - c_a}{v_a}$. Equating these two expressions and solving for v_b in terms of v_a, we have

$$v_b = \frac{pv_a}{-c_a + (1-\delta)p + v_a(1 - \pi_a\alpha_a)}. \tag{A.32}$$

Plugging this expression for v_b into (A.30), we find that v_a must be a zero of the cubic equation:

$$f_2(x) \triangleq c_a^2 - 2c_a(1-\delta)p + ((1-\delta)p)^2 + x(c_a - (1-\delta)p)(-1 + 2\pi_a\alpha_a) + \\ x^3\pi_s\alpha_s(1 - \pi_a\alpha_a) - x^2(\pi_a\alpha_a(1 - \pi_a\alpha_a) + \pi_s\alpha_s(c_a + p\delta)). \tag{A.33}$$

To find which root of the cubic v_b must be, first note that $\pi_a\alpha_a - \pi_s\alpha_s u < 0$ for consumers of higher valuation to prefer (B, NP) over (B, AP) by (A.13). From that, we have that $c_a - (1-\delta)p > 0$ in order for $v_a > 0$. To pin down the root of the cubic, note that the cubic's highest order term is $\pi_s\alpha_s(1 - \pi_a\alpha_a)x^3$, so $\lim_{x\to-\infty} f_2(x) = -\infty$ and $\lim_{x\to\infty} f_2(x) = \infty$. We find $f_2(0) = ((1-\delta)p - c_a)^2 > 0$ and $f_2\left(\frac{c_a - (1-\delta)p}{1 - \pi_a\alpha_a}\right) = -\frac{p\pi_s\alpha_s((1-\delta)p - c_a)^2}{(1 - \pi_a\alpha_a)^2} < 0$.

We note that from (A.30), it follows that $1 - \pi_s\alpha_s(v_a - v_b) > 0$. It follows that $\pi_s\alpha_s(v_a - v_b) < 1$, so that $\frac{c_a - (1-\delta)p}{\pi_s\alpha_s(v_a - v_b) - \pi_a\alpha_a} > \frac{c_a - (1-\delta)p}{1 - \pi_a\alpha_a}$. Hence, by (A.31), we have that $v_a > \frac{c_a - (1-\delta)p}{1 - \pi_a\alpha_a}$. Therefore, v_a is the largest root of the cubic, lying past $\frac{c_a - (1-\delta)p}{1 - \pi_a\alpha_a}$. Then using (A.32), we solve for v_b.

For this to be an equilibrium, the necessary and sufficient conditions are $0 < v_b < v_a < 1$ and type $v = 1$ prefers (B, AP) over (B, P). Type $v = 1$ preferring (B, AP) over (B, P) ensures $v < 1$ does so too, by (A.11). Moreover, since type $v = v_a$ is indifferent between (B, AP) and (B, NP), and since (B, AP) is preferred over (B, P), by transitivity, it follows that type v_a prefers (B, NP) over (B, P). It follows that $v < v_a$ prefers (B, NP) over (B, P) as well by (A.11).

For $v_a < 1$, an equivalent condition for this to hold is $f_2(1) > 0$, so that the third root of $f_2(x)$ is less than 1. Since $(1-\delta)p - c_a < 0$ (again, from $v_b > 0$), this condition simplifies to $c_a^2 + ((1-\delta)p)^2 + (1-\delta)p(1 - 2\pi_a\alpha_a) - (1 - \pi_a\alpha_a)(\pi_a\alpha_a - \pi_s\alpha_s) - c_a(1 + 2(1-\delta)p - 2\pi_a\alpha_a + \pi_s\alpha_s) > \pi_s\alpha_s\delta p$.

For $v_a > v_b$, using (A.32), it is equivalent to require $v_a > \frac{c_a + p\delta}{1 - \pi_a\alpha_a}$. Note that $\frac{c_a + p\delta}{1 - \pi_a\alpha_a} > \frac{c_a - (1-\delta)p}{1 - \pi_a\alpha_a}$, so for this to happen, we need the condition $f_2(\frac{c_a + p\delta}{1 - \pi_a\alpha_a}) < 0$, which simplifies to $(1-\delta)p - c_a < p\pi_a\alpha_a$. But this condition is already implied by the condition $(1-\delta)p - c_a < 0$.

For $v_b > 0$, using (A.32), a necessary and sufficient condition for this to hold is $v_a > \frac{c_a - (1-\delta)p}{1 - \pi_a\alpha_a}$. But this holds by construction of v_a as the largest root of $f_2(x)$, so no additional conditions are needed.

Finally, for type $v = 1$ to prefer (B, AP) over (B, P), a necessary and sufficient condition is $\pi_a\alpha_a \leq (1-\delta)p + c_p - c_a$. The conditions above are summarized in (A.29). $\square$

Next, for case (VI), in which all segments are represented and the middle tier does automated patching, i.e., $0 < v_b < v_a < v_p < 1$, we have $u = v_a - v_b$. Following the same steps as before, we prove the following claim related to the corresponding parameter region in which case (VI) arises.

Claim 6 *The equilibrium that corresponds to case (VI) arises if and only if the following conditions are satisfied:*

$$(1-\delta)p < c_a \text{ and } c_p + (1-\delta)p < c_a + \pi_a\alpha_a \text{ and } c_p(1-\pi_a\alpha_a) + p(1-\delta) > c_a \text{ and}$$
$$c_p(\pi_a\alpha_a)^2(-c_a + c_p(1-\pi_a\alpha_a) + p(1-\delta)) +$$
$$\pi_s\alpha_s(-c_a + c_p + p(1-\delta))^2(c_a - c_p(1-\pi_a\alpha_a) - (1-\delta-\pi_a\alpha_a)p) < 0. \quad \text{(A.34)}$$

To solve for the thresholds v_b and v_a, using $u = v_a - v_b$, note that they solve

$$v_b = \frac{p}{1 - \pi_s\alpha_s(v_a - v_b)}, \text{ and} \quad \text{(A.35)}$$

$$v_a = \frac{(1-\delta)p - c_a}{\pi_a\alpha_a - \pi_s\alpha_s(v_a - v_b)}. \quad \text{(A.36)}$$

These are the same as (A.30) and (A.31). Using the exact same argument, it follows that v_b is the largest root of the cubic $f_2(x)$, lying past $\frac{c_a-(1-\delta)p}{1-\pi_a\alpha_a}$. Then using (A.32), we solve for v_b.

In this case, however, we also have a patching population, with the patching threshold given by $v_p = \frac{c_p-(c_a-(1-\delta)p)}{\pi_a\alpha_a}$.

For this to be an equilibrium, the necessary and sufficient conditions are $0 < v_b < v_a < v_p < 1$. This tells us that all $v \in [v_p, 1]$ have the same preferences and will purchase (B, P), all $v \in [v_a, v_p)$ have the same preferences and will purchase (B, AP), and all $v \in [v_b, v_a)$ have the same preferences and will purchase (B, NP). Finally, all $v < v_b$ have the same preferences and will not purchase in equilibrium.

For $v_p < 1$, the necessary and sufficient condition is $c_p + (1-\delta)p < c_a + \pi_a\alpha_a$.

For $v_a < v_p$, it is equivalent to write $f_2\left(\frac{c_p-(c_a-(1-\delta)p)}{\pi_a\alpha_a}\right) > 0$. Simplifying, this gives the condition $c_p(\pi_a\alpha_a)^2(-c_a + c_p(1-\pi_a\alpha_a) + p(1-\delta)) + \pi_s\alpha_s(-c_a + c_p + p(1-\delta))^2(c_a - c_p(1-\pi_a\alpha_a) - (1-\delta-\pi_a\alpha_a)p) < 0$.

For $v_b < v_a$, the necessary and sufficient condition reduces down to $v_a > \frac{c_a+\delta p}{1-\pi_a\alpha_a}$. As in the previous case, this condition isn't necessary given $(1-\delta)p - c_a < 0$.

For $0 < v_b$, it is equivalent to have $v_a > \frac{c_a-p(1-\delta)}{1-\pi_a\alpha_a}$, which holds by construction of v_a.

A summary of the above necessary and sufficient conditions is given in (A.34). □

Next, for case (VII), in which there are no automated patching users while the lower tier in unpatched and the upper tier is patched, i.e., $0 < v_b < v_p < 1$, we have $u = v_p - v_b$. Following the same steps as before, we prove the following claim related to the corresponding parameter region in which case (VII) arises.

Claim 7 *The equilibrium that corresponds to case (VII) arises if and only if the following condi-*

tions are satisfied:

$$\begin{aligned}
&c_a > (1-\delta-\pi_a\alpha_a)p \text{ and } c_p^2 > c_p - (1-c_p-p)\pi_s\alpha_s \text{ and} \\
&(1-\pi_a\alpha_a)(c_a-(1-\delta-\pi_a\alpha_a)p)^2 + \pi_s\alpha_s(c_a+\delta p)^2(c_a - c_p(1-\pi_a\alpha_a) - (1-\delta-\pi_a\alpha_a)p) \geq 0 \text{ and} \\
&\Bigg\{\Big(c_a - (1-\delta)p - c_p(1-\pi_a\alpha_a) \geq 0\Big) \text{ or} \\
&\Big(c_a - (1-\delta)p - c_p(1-\pi_a\alpha_a) < 0 \text{ and } c_p(1-\pi_a\alpha_a) + (1-\delta)p > c_a \text{ and} \\
&c_p(\pi_a\alpha_a)^2(-c_a+c_p(1-\pi_a\alpha_a)+p(1-\delta))+\pi_s\alpha_s(-c_a+c_p+p(1-\delta))^2(c_a-c_p(1-\pi_a\alpha_a)-(1-\delta-\pi_a\alpha_a)p) \geq 0\Big)\Bigg\}.
\end{aligned} \tag{A.37}$$

To solve for the thresholds v_b and v_p, using $u = v_p - v_b$, note that they solve

$$v_b = \frac{p}{1-\pi_s\alpha_s(v_p - v_b)}, \text{ and} \tag{A.38}$$

$$v_p = \frac{c_p}{\pi_s\alpha_s(v_p - v_b)}. \tag{A.39}$$

From (A.38), we have $\pi_s\alpha_s(v_p - v_b) = 1 - \frac{p}{v_b}$, while from (A.39), we have $\pi_s\alpha_s(v_p - v_b) = \frac{c_p}{v_p}$. Equating these two expressions and solving for v_p in terms of v_b, we have

$$v_p = \frac{c_p v_b}{v_b - p}. \tag{A.40}$$

Plugging this expression for v_p into (A.38), we find that v_b must be a zero of the cubic equation:

$$f_3(x) \triangleq p^2 + x^2(1 - c_p\pi_s\alpha_s + \pi_s\alpha_s x) - px(2 + \pi_s\alpha_s x). \tag{A.41}$$

To find which root of the cubic v_b must be, note that the cubic's highest order term is $\pi_s\alpha_s x^3$, so $\lim_{x\to-\infty} f_3(x) = -\infty$ and $\lim_{x\to\infty} f_3(x) = \infty$. We find $f_3(0) = p^2 > 0$, $f_3(p) = -c_p\pi_s\alpha_s p^2 < 0$, and $f_3(c_p+p) = c_p^2 > 0$. Since $v_b - p > 0$ in equilibrium, we have that v_b is the largest root of the cubic, lying past p. Then using (A.40), we solve for v_p.

For this to be an equilibrium, the necessary and sufficient conditions are $0 < v_b < v_p < 1$, type $v = v_p$ prefers (B, P) over (B, AP), and type $v = v_b$ prefers (NB, NP) to (B, AP). Type $v = v_p$ preferring (B, P) over (B, AP) ensures $v > v_p$ also prefer (B, P) over (B, AP), by (A.11). Moreover, type $v = v_b$ preferring (NB, NP) over (B, AP) ensures $v < v_b$ do so too, by (A.12).

For $v_p < 1$, a necessary and sufficient condition for this to hold is $v_b > \frac{p}{1-c_p}$. This is equivalent to $f_3(\frac{p}{1-c_p}) < 0$. Using $v_b < c_p + p$ and the parameter assumptions of the model, this simplifies to $c_p^2 > c_p - (1 - c_p - p)\pi_s\alpha_s$.

For $v_p > v_b > 0$, a necessary and sufficient condition is $0 < v_b < c_p + p$, which is true by construction of v_b.

For type $v = v_p$ to prefer (B, P) over (B, AP), a necessary and sufficient condition is $\frac{c_p v_b}{v_b - p} \geq \frac{c_p - (c_a - (1-\delta)p)}{\pi_a\alpha_a}$. This simplifies to $v_b(c_a - (1-\delta)p - c_p(1-\pi_a\alpha_a)) \geq p(c_a - (1-\delta)p - c_p)$. This can be broken down into three cases, depending on the sign of $c_a - (1-\delta)p - c_p(1-\pi_a\alpha_a)$ (also considering the case when the factor is zero). When $c_a-(1-\delta)p-c_p(1-\pi_a\alpha_a)=0$, the left side is 0 while the right side is negative, so this inequality holds. If $c_a-(1-\delta)p-c_p(1-\pi_a\alpha_a)>0$, then the inequality becomes $v_b\geq p\frac{c_a-(1-\delta)p-c_p}{c_a-(1-\delta)p-c_p(1-\pi_a\alpha_a)}$. But $\frac{c_a-(1-\delta)p-c_p}{c_a-(1-\delta)p-c_p(1-\pi_a\alpha_a)}<1$, and since $v_b>p$ by construction, this inequality holds without further conditions. On the other hand, if $c_a-(1-\delta)p-c_p(1-\pi_a\alpha_a)<0$, then we need $v_b\leq p\frac{c_a-(1-\delta)p-c_p}{c_a-(1-\delta)p-c_p(1-\pi_a\alpha_a)}$. So we need $f_3\left(\frac{p(c_a-(1-\delta)p-c_p)}{c_a-(1-\delta)p-c_p(1-\pi_a\alpha_a)}\right)\geq 0$. Omitting the algebra, this simplifies to $c_p(1-\pi_a\alpha_a)+(1-\delta)p>c_a$ and $c_p(\pi_a\alpha_a)^2(-c_a+c_p(1-\pi_a\alpha_a)+p(1-\delta))+\pi_s\alpha_s(-c_a+c_p+p(1-\delta))^2(c_a-c_p(1-\pi_a\alpha_a)-(1-\delta-\pi_a\alpha_a)p)\geq 0$.

For $v=v_b$ to prefer (NB,NP) to (B,AP), a necessary and sufficient condition is $v_b\leq\frac{\delta p+c_a}{1-\pi_a\alpha_a}$, which becomes $f_3\left(\frac{\delta p+c_a}{1-\pi_a\alpha_a}\right)\geq 0$. This simplifies to $c_a>(1-\delta-\pi_a\alpha_a)p$ and $(1-\pi_a\alpha_a)(c_a-(1-\delta-\pi_a\alpha_a)p)^2+\pi_s\alpha_s(c_a+\delta p)^2(c_a-c_p(1-\pi_a\alpha_a)-(1-\delta-\pi_a\alpha_a)p)\geq 0$. The conditions are summarized in (A.37). $\square$

This completes the proof of the general consumer market equilibrium for the proprietary case. $\square$

Proof of Lemma 3: Technically, we prove that there exists an α_1 such that for $\alpha_s>\alpha_1$, p^* and δ^* are set so that

1. if $c_a<\min\left[\pi_a\alpha_a-c_p,c_p(1-\pi_a\alpha_a)\right]$, then $\sigma^*(v)$ is characterized by $0<v_a<v_b<v_p<1$ under optimal pricing,
2. if $|\pi_a\alpha_a-c_p|<c_a<c_p(1-\pi_a\alpha_a)$, then $\sigma^*(v)$ is characterized by $0<v_b<v_a<v_p<1$ under optimal pricing, and if
3. if $c_a>c_p(1-\pi_a\alpha_a)$, then $\sigma^*(v)$ is characterized by $0<v_b<v_p<1$ under optimal pricing.

Suppose that $0<v_b<1$. By part (IV) of Lemma A.2, we obtain $v_b=1-\frac{1-p}{\pi_s\alpha_s}+\frac{(1-p)p}{(\pi_s\alpha_s)^2}+O\left(\frac{1}{(\pi_s\alpha_s)^3}\right)$, which upon substitution into (6) yields

$$\Pi(p,\delta)=\frac{1-2p}{\pi_s\alpha_s}-\frac{p(2-3p)}{(\pi_s\alpha_s)^2}-\frac{p(2-(9-8p)p)}{(\pi_s\alpha_s)^3}+O\left(\frac{1}{(\pi_s\alpha_s)^4}\right). \tag{A.42}$$

Its unconstrained maximizer is given by

$$p_N=\frac{1}{2}-\frac{1}{8\pi_s\alpha_s}+\frac{1}{16(\pi_s\alpha_s)^2}+O\left(\frac{1}{(\pi_s\alpha_s)^3}\right). \tag{A.43}$$

Substituting (A.43) into (A.42), we obtain

$$\Pi_N=\frac{1}{4\pi_s\alpha_s}-\frac{1}{8(\pi_s\alpha_s)^2}+O\left(\frac{1}{(\pi_s\alpha_s)^3}\right). \tag{A.44}$$

Suppose that $0<v_a<v_b<v_p<1$ is induced. By part (III) of Lemma A.2, we obtain $v_a=\frac{\delta p+c_a}{1-\pi_a\alpha_a}$ and $v_b=\frac{c_p-c_a+(1-\delta)p}{\pi_a\alpha_a}-\frac{c_p^2\pi_a\alpha_a}{(c_p-c_a+(1-\delta)p)^2\pi_s\alpha_s}+O\left(\frac{1}{(\pi_s\alpha_s)^2}\right)$, which upon substitution into (6) yields

$$\Pi(p,\delta)=p\left(1+\frac{(c_a-c_p-(1-\delta)p)(1-\delta)}{\pi_a\alpha_a}-\frac{\delta(c_a+p\delta)}{1-\pi_a\alpha_a}\right)+\frac{c_p^2p\pi_a\alpha_a(1-\delta)}{c_a-c_p-(1-\delta)p)^2\pi_s\alpha_s}+O\left(\frac{1}{(\pi_s\alpha_s)^2}\right). \tag{A.45}$$

Its unconstrained maximizers are given by

$$p_A = \frac{1-c_p}{2} + \frac{2c_p^2(\pi_a\alpha_a)^2(3(c_a-c_p)+\pi_a\alpha_a)}{(c_a-c_p-\pi_a\alpha_a)^3\pi_s\alpha_s} + O\left(\frac{1}{(\pi_s\alpha_s)^2}\right), \text{ and} \tag{A.46}$$

$$\delta_A = \frac{1-\pi_a\alpha_a-c_a}{1-c_p} - \frac{4c_p^2(\pi_a\alpha_a)^2(1-c_a-\pi_a\alpha_a)(3(c_a-c_p)+\pi_a\alpha_a)}{(1-c_p)^2(c_a-c_p-\pi_a\alpha_a)^3\pi_s\alpha_s} + O\left(\frac{1}{(\pi_s\alpha_s)^2}\right). \tag{A.47}$$

Substituting (A.46) and (A.47) into (A.45), we obtain

$$\Pi_A = \frac{1}{4}\left(1-2c_p+\frac{(c_a-c_p)^2}{\pi_a\alpha_a}+\frac{c_a^2}{1-\pi_a\alpha_a}\right) + \frac{2c_p^2\pi_a\alpha_a(c_a-c_p+\pi_a\alpha_a)}{(c_a-c_p-\pi_a\alpha_a)^2\pi_s\alpha_s} + O\left(\frac{1}{(\pi_s\alpha_s)^2}\right). \tag{A.48}$$

Suppose that $0<v_b<v_a<v_p<1$ is induced. By part (VI) of Lemma A.2, we obtain $v_a = \frac{c_a+p\delta}{1-\pi_a\alpha_a} + \frac{p(c_a-p(1-\delta-\pi_a\alpha_a))}{(c_a+p\delta)^2\pi_s\alpha_s} + O\left(\frac{1}{(\pi_s\alpha_s)^2}\right)$, $v_b = \frac{c_a+p\delta}{1-\pi_a\alpha_a} - \frac{(c_a-(1-\delta)p)(c_a-(1-\delta-\pi_a\alpha_a)p}{(c_a+p\delta)^2\pi_s\alpha_s} + O\left(\frac{1}{(\pi_s\alpha_s)^2}\right)$, and $v_p = \frac{c_p-(c_a-(1-\delta)p)}{\pi_a\alpha_a}$, which upon substitution into (6) yields

$$\Pi(p,\delta) = \frac{p\left(c_a+c_p(-1+\delta)+p(-1+2\delta)+\pi_a\alpha_a+\frac{\delta(c_a+p\delta)}{-1+\pi_a\alpha_a}\right)}{\pi_a\alpha_a} + \frac{c_a p(c_a+p(-1+\delta+\pi_a\alpha_a))}{(c_a+p\delta)^2\pi_s\alpha_s} + O\left(\frac{1}{(\pi_s\alpha_s)^2}\right). \tag{A.49}$$

Its unconstrained maximizers are given by

$$p_B = \frac{1-c_p}{2} + \frac{c_a(-1+c_a+2c_p+\pi_a\alpha_a-2c_p\pi_a\alpha_a)(c_a+c_p-c_p\pi_a\alpha_a)}{(1+c_a-\pi_a\alpha_a)^3\pi_s\alpha_s} + O\left(\frac{1}{(\pi_s\alpha_s)^2}\right), \text{ and} \tag{A.50}$$

$$\delta_B = \frac{1-c_a-\pi_a\alpha_a}{1-c_p} + \frac{2c_a(-1+c_a+2c_p+\pi_a\alpha_a-2c_p\pi_a\alpha_a)\left(c_a^2+(-1+\pi_a\alpha_a)\left(c_p^2+\pi_a\alpha_a-2c_p\pi_a\alpha_a\right)\right)}{(1-c_p)^2(1+c_a-\pi_a\alpha_a)^3\pi_s\alpha_s} + O\left(\frac{1}{(\pi_s\alpha_s)^2}\right). \tag{A.51}$$

Substituting (A.50) and (A.51) into (A.49), we obtain

$$\Pi_B = \frac{1}{4}\left(1-2c_p+\frac{(c_a-c_p)^2}{\pi_a\alpha_a}+\frac{c_a^2}{1-\pi_a\alpha_a}\right) - \frac{c_a(-1+c_p)(c_a+c_p-c_p\pi_a\alpha_a)}{(1+c_a-\pi_a\alpha_a)^2\pi_s\alpha_s} + O\left(\frac{1}{(\pi_s\alpha_s)^2}\right). \tag{A.52}$$

Suppose that $0<v_b<v_p<1$ is induced. By part (VII) of Lemma A.2, we obtain $v_b = c_p+p-\frac{c_p^2}{(c_p+p)^2\pi_s\alpha_s} + O\left(\frac{1}{(\pi_s\alpha_s)^2}\right)$, which upon substitution into (6) yields

$$\Pi(p,\delta) = p(1-c_p-p) + \frac{c_p^2 p}{(c_p+p)^2\pi_s\alpha_s} - \frac{2c_p^3p^2}{(c_p+p)^5(\pi_s\alpha_s)^2} + O\left(\frac{1}{(\pi_s\alpha_s)^3}\right). \tag{A.53}$$

Its unconstrained maximizer is given by

$$p_P = \frac{1-c_p}{2} - \frac{2c_p^2(1-3c_p)}{(1+c_p)^3\pi_s\alpha_s} - \frac{16c_p^3(-3+8c_p-5c_p^2+8c_p^3)}{(1+c_p)^7(\pi_s\alpha_s)^2} + O\left(\frac{1}{(\pi_s\alpha_s)^3}\right). \tag{A.54}$$

Substituting (A.54) into (A.53), we obtain

$$\Pi_P = \frac{1}{4}(1-c_p)^2 + \frac{2(1-c_p)c_p^2}{(1+c_p)^2\pi_s\alpha_s} + O\left(\frac{1}{(\pi_s\alpha_s)^2}\right). \tag{A.55}$$

Finally, suppose that $0 < v_a < 1$ is induced in equilibrium. By part (I) of Lemma A.2, we obtain $v_a = \frac{\delta p + c_a}{1-\pi_a\alpha_a}$, which upon substitution into (6) yields

$$\Pi_a = \frac{(1-c_a-\pi_a\alpha_a)^2}{4(1-\pi_a\alpha_a)}. \tag{A.56}$$

The unconstrained profits under $0 < v_a < 1$, $0 < v_a < v_b < 1$, and $0 < v_b < v_a < 1$ differ by $O\left(\frac{1}{(\pi_s\alpha_s)}\right)$. Using the case for $0 < v_a < 1$, we show below that all outcomes are dominated by one of the above cases under the conditions of the lemma.

Comparing (A.44) with (A.48), (A.52), or (A.55), we find that (A.44) is dominated by the other two profits when α_s is sufficiently large (say, $\alpha_s > \underline{\alpha}_1$, for some $\underline{\alpha}_1$). Using (A.46) and (A.47) with Lemma A.2, we find that, for sufficiently large α_s ($\alpha_s > \underline{\alpha}_2$, for some $\underline{\alpha}_2$), the optimal price of this case indeed induces the correct market structure of $0 < v_a < v_b < v_p < 1$ when $c_a < \min\left[\pi_a\alpha_a\left(1 - \frac{1}{2-\pi_a\alpha_a}\right), \pi_a\alpha_a - c_p, c_p(1-\pi_a\alpha_a)\right]$. We can simplify this since $c_a < \pi_a\alpha_a\left(1 - \frac{1}{2-\pi_a\alpha_a}\right)$ holds whenever $c_a < \min\left[\pi_a\alpha_a - c_p, c_p(1-\pi_a\alpha_a)\right]$. Then using (A.50) and (A.51) with Lemma A.2, we find that, for sufficiently large α_s ($\alpha_s > \underline{\alpha}_3$, for some $\underline{\alpha}_3$), the optimal price of this case indeed induces the correct market structure when $|\pi_a\alpha_a - c_p| < c_a < c_p(1-\pi_a\alpha_a)$. Similarly, using (A.54) with Lemma A.2, we find that, for sufficiently large α_s ($\alpha_s > \underline{\alpha}_4$, for some $\underline{\alpha}_4$), the optimal price of this case indeed induces the correct market structure when $c_a > c_p - \frac{1}{2}(1+c_p)\pi_a\alpha_a$. Let α_1 be the max of these $\underline{\alpha}_1$, $\underline{\alpha}_2$, and $\underline{\alpha}_3$. Then for $\alpha_s > \alpha_1$, if $c_a > c_p - \pi_a\alpha_a$, we have that (A.44) is dominated by the other profits, at least one of which has its profit-maximizing solution inducing the market structure it represents.

Comparing (A.48) with the other profits, we find that if its interior solution induces the conjectured market structure of $0 < v_a < v_b < v_p < 1$, then it dominates the other profits when $\alpha_s > \alpha_1$. Comparing (A.52) with the other profits, we find that if its interior solution induces the conjectured market structure of $0 < v_a < v_b < v_p < 1$, then it dominates the other profits when $\alpha_s > \alpha_1$. Comparing (A.55) with (A.48) and (A.52), we see that (A.55) never dominates when either of the other two market structures can be induced with an interior solution. Comparing it with the unconstrained maximal profit of $0 < v_a < 1$, $0 < v_b < v_p < 1$ has greater profit when $2c_a(1-\pi_a\alpha_a) > c_a^2 + (1-\pi_a\alpha_a)(c_p(2-c_p) - \pi_a\alpha_a)$, which can be written as $c_a > 1 - \pi_a\alpha_a - (1-c_p)\sqrt{1-\pi_a\alpha_a}$. This is also the condition that ensures $0 < v_b < v_p < 1$ has greater profit than $0 < v_b < v_a < 1$ and $0 < v_a < v_b < 1$. Under this condition, (A.55) also beats the profit at the boundary of $0 < v_a < v_b < v_p < 1$ and $0 < v_b < v_a < v_p < 1$. Since this boundary profit beats any other boundary profit, it follows that $c_a > 1 - \pi_a\alpha_a - (1-c_p)\sqrt{1-\pi_a\alpha_a}$ is the only condition we need for $0 < v_b < v_p < 1$ to beat the profits at the boundaries between cases. Then since

$c_p(1-\pi_a\alpha_a) > \max(c_p-\frac{1}{2}(1+c_p)\pi_a\alpha_a, 1-\pi_a\alpha_a-(1-c_p)\sqrt{1-\pi_a\alpha_a})$, it follows that $c_a \geq c_p(1-\pi_a\alpha_a)$. This is because it dominates the maximal profit of $0<v_a<1$ and any boundary case profits while not being dominated by $0<v_a<v_b<v_p<1$ or $0<v_b<v_a<v_p<1$. Hence, $0<v_b<v_p<1$ is induced in equilibrium. □

3 Proofs of Propositions

Proof of Proposition 1: We focus on the region in which all segments are represented under optimal pricing in the base case. Specifically, for sufficiently high α_s, by Lemma 2, we have that p^* is set so that if $c_p - \pi_a\alpha_a < c_a < 1 - \pi_a\alpha_a - (1-c_p)\sqrt{1-\pi_a\alpha_a}$, then $\sigma^*(v)$ is characterized by $0<v_b<v_a<v_p<1$ under optimal pricing. By Lemma 3, when patching rights are priced under the same parameter region, there are two cases: either $0<v_a<v_b<v_p<1$ is induced or $0<v_b<v_a<v_p<1$ is induced. Specifically, p^* and δ^* are set so that

(i) if $c_a < \min[\pi_a\alpha_a - c_p, c_p(1-\pi_a\alpha_a)]$, then $\sigma^*(v)$ is characterized by $0<v_a<v_b<v_p<1$ under optimal pricing, and

(ii) if $|\pi_a\alpha_a - c_p| < c_a < c_p(1-\pi_a\alpha_a)$, then $\sigma^*(v)$ is characterized by $0<v_b<v_a<v_p<1$ under optimal pricing.

In either case, since $c_p(1-\pi_a\alpha_a) > 1 - \pi_a\alpha_a - (1-c_p)\sqrt{1-\pi_a\alpha_a}$ using the assumptions that $0 < c_p < 1$ and $0 < \pi_a\alpha_a < 1$, we have that $c_p - \pi_a\alpha_a < c_a < 1 - \pi_a\alpha_a - (1-c_p)\sqrt{1-\pi_a\alpha_a}$ is a subset of the union of the regions $c_a < \min[\pi_a\alpha_a - c_p, c_p(1-\pi_a\alpha_a)]$ and $|\pi_a\alpha_a - c_p| < c_a < c_p(1-\pi_a\alpha_a)$. Moreover, the intersection of $c_p - \pi_a\alpha_a < c_a < 1 - \pi_a\alpha_a - (1-c_p)\sqrt{1-\pi_a\alpha_a}$ with either $c_a < \min[\pi_a\alpha_a - c_p, c_p(1-\pi_a\alpha_a)]$ or $|\pi_a\alpha_a - c_p| < c_a < c_p(1-\pi_a\alpha_a)$ is non-empty.

In the first case, the optimal profit in the base case when patching rights aren't priced is given by (A.6), and optimal profit when patching rights are priced is given by (A.48). The percentage increase in profit is given by $\frac{(1-\pi_a\alpha_a)(c_a-c_p+\pi_a\alpha_a)^2}{\pi_a\alpha_a(1-c_a-\pi_a\alpha_a)^2} + O\left(\frac{1}{\pi_s\alpha_s}\right)$. Moreover, the size of the unpatched population when patching rights arent priced is given by In the first case, the optimal profit in the base case when patching rights aren't priced is given by (A.6), and optimal profit when patching rights are priced is given by (A.48). The percentage increase in profit is given by $\frac{(1-\pi_a\alpha_a)(c_a-c_p+\pi_a\alpha_a)^2}{\pi_a\alpha_a(1-c_a-\pi_a\alpha_a)^2} + O\left(\frac{1}{\pi_s\alpha_s}\right)$. Moreover, the size of the unpatched population when patching rights arent priced is given by

$$\hat{u}(\sigma^*|\text{patching rights not priced}) = \frac{\pi_a\alpha_a(1-\pi_a\alpha_a) + c_a(2-\pi_a\alpha_a)}{(1+c_a-\pi_a\alpha_a)\pi_s\alpha_s} + O\left(\frac{1}{(\pi_s\alpha_s)^2}\right), \tag{A.57}$$

while it shrinks to

$$\tilde{u}_1(\sigma^*|\text{patching rights priced}) = \frac{2c_p\pi_a\alpha_a}{(c_p-c_a+\pi_a\alpha_a)\pi_s\alpha_s} + O\left(\frac{1}{(\pi_s\alpha_s)^2}\right) \tag{A.58}$$

when patching rights are priced. That (A.58) is smaller than (A.57) follows from $0 < c_a < c_p < 1$ and $\frac{-1+c_a+\pi_a\alpha_a+\sqrt{1-\pi_a\alpha_a}}{\sqrt{1-\pi_a\alpha_a}} < c_p < \pi_a\alpha_a - c_a$, which are conditions in this parameter region.

Similarly, in the second case, the optimal profit in the base case is again given by (A.6), and optimal profit when patching rights are priced is given by (A.52). The percentage increase in profit is again $\frac{(1-\pi_a\alpha_a)(c_a-c_p+\pi_a\alpha_a)^2}{\pi_a\alpha_a(1-c_a-\pi_a\alpha_a)^2} + O\left(\frac{1}{\pi_s\alpha_s}\right)$. Moreover, the size of the unpatched population when patching rights arent priced is given by (A.57), while it shrinks to

$$\tilde{u}_2(\sigma^*|\text{patching rights priced}) = \frac{c_a + c_p(1-\pi_a\alpha_a)}{(1+c_a-\pi_a\alpha_a)\pi_s\alpha_s} + O\left(\frac{1}{(\pi_s\alpha_s)^2}\right) \tag{A.59}$$

when patching rights are priced. That (A.59) is smaller than (A.57) follows from $0 < c_a < 1$, $0 < \pi_a\alpha_a < 1$, and $c_a > c_p - \pi_a\alpha_a$, which are conditions in this parameter region.

In either case, we have that pricing patching rights increases profits while reducing the size of the unpatched population in equilibrium as compared to the base case when patching rights aren't priced. ■

Proof of Proposition 2: Again, when patching rights are priced, there are two cases: either $0 < v_a < v_b < v_p < 1$ is induced or $0 < v_b < v_a < v_p < 1$ is induced in equilibrium. In either case, it suffices to show that the price of the automated patching option is higher when patching rights are priced than in the base case when automated patching costs are low enough.

In the first region, the price of the automated patching option when patching rights arent priced is given in (A.5). Using (A.46) and (A.47), when patching rights are priced, the automated patching option is priced at

$$\delta^* p^* = \frac{1}{2}(1 - c_a - \pi_a\alpha_a) + O\left(\frac{1}{(\pi_s\alpha_s)^2}\right). \tag{A.60}$$

Comparing the two, we have that $\delta^* p^*$ when patching rights are priced is greater than the price set in the base case when $c_a < \frac{1}{3-2\pi_a\alpha_a} - \pi_a\alpha_a$. The intersection of this with the parameter region of this case is non-empty when $\pi_a\alpha_a < \frac{1}{2}$.

The argument for the second case is similar and is omitted for brevity, and we find (for the case when $0 < v_b < v_a < v_p < 1$ is induced in equilibrium) that $\delta^* p^*$ is greater than the price of the base case if $c_a < \frac{(1-2c_p)(1-\pi_a\alpha_a)}{5-4\pi_a\alpha_a}$. The intersection of this with the parameter region of this case is non-empty also when $\pi_a\alpha_a < \frac{1}{2}$. ■

Proof of Proposition 3: Using Lemma 2, the first part of Proposition 3 has status quo pricing inducing $0 < v_b < v_a < v_p < 1$ market structure wihile the second part of Proposition 3 has status quo pricing inducing $0 < v_b < v_p < 1$. Using Lemma 3, both parts of Proposition 3 have the vendor inducing either $0 < v_a < v_b < v_p < 1$ or $0 < v_b < v_a < v_p < 1$ under priced patching rights. Using the definition of social welfare in 20 , we find that the social welfare under status quo pricing of the first part of Proposition 3 is given by

$$W_{SQ_1} = \frac{1}{8}\left(3 + 2c_a - 8c_p + \frac{4(c_a-c_p)^2}{\pi_a\alpha_a} + \pi_a\alpha_a + \frac{3c_a^2}{1-\pi_a\alpha_a}\right) + O\left(\frac{1}{\pi_s\alpha_s}\right), \tag{A.61}$$

and the social welfare under priced patching rights is given by

$$W_{PP} = \frac{3}{8}\left(1 - 2c_p + \frac{(c_a-c_p)^2}{\pi_a\alpha_a} + \frac{c_a^2}{1-\pi_a\alpha_a}\right) + O\left(\frac{1}{\pi_s\alpha_s}\right). \tag{A.62}$$

The difference is

$$W_{SQ_1} - W_{PP} = \frac{(c_a - c_p + \pi_a\alpha_a)^2}{8\pi_a\alpha_a} + O\left(\frac{1}{\pi_s\alpha_s}\right), \tag{A.63}$$

establishing the first part of Proposition 3.

For the second half of Proposition 3, the social welfare under status quo pricing is given by

$$W_{SQ_2} = \frac{3}{8}(1 - c_p)^2 + O\left(\frac{1}{\pi_s\alpha_s}\right). \tag{A.64}$$

Comparing with A.62, we have that the difference is

$$W_{PP} - W_{SQ_2} = \frac{3(c_a - c_p(1 - \pi_a\alpha_a))^2}{8(\pi_a\alpha_a(1 - \pi_a\alpha_a))} + O\left(\frac{1}{\pi_s\alpha_s}\right), \tag{A.65}$$

establishing the second part of Proposition 3. ■

4 Case: Open-Source Software

Lemma A.3 *The consumer market equilibrium for an open-source software product made available for free is given by the following:*

(I) If either (i) $\pi_a\alpha_a > c_p - c_a$ *and* $\pi_s\alpha_s \le c_p$*, or (ii)* $\pi_a\alpha_a \le c_p - c_a$ *and* $\pi_s\alpha_s \le c_a + \pi_a\alpha_a$*, then* $0 = v_b < v_a = v_p = 1$ *;*

(II) If either (i) $\pi_a\alpha_a \ge 1 - c_a/c_p$ *and* $c_p < \pi_s\alpha_s \le 1/c_p$*, or (ii)* $c_p - c_a < \pi_a\alpha_a < 1 - c_a/c_p$ *and* $c_p < \pi_s\alpha_s \le \frac{c_p(\pi_a\alpha_a)^2}{(c_p - c_a)^2}$*, then* $0 = v_b < v_a = v_p < 1$*, where* $v_p = \sqrt{\frac{c_p}{\pi_s\alpha_s}}$ *;*

(III) If $\pi_a\alpha_a \ge 1 - c_a/c_p$ *and* $\pi_s\alpha_s > 1/c_p$*, then* $0 < v_b < v_a = v_p < 1$*, where* $v_b = c_p - 1/\pi_s\alpha_s$ *and* $v_p = c_p$ *;*

(IV) If $c_p - c_a < \pi_a\alpha_a < 1 - c_a/c_p$ *and* $\pi_s\alpha_s > \frac{1 - \pi_a\alpha_a}{c_a}$*, then* $0 < v_b < v_a < v_p < 1$*, where* $v_b = \frac{c_a}{1 - \pi_a\alpha_a} - \frac{1}{\pi_s\alpha_s}$*,* $v_a = \frac{c_a}{1 - \pi_a\alpha_a}$ *and* $v_p = \frac{c_p - c_a}{\pi_a\alpha_a}$ *;*

(V) If $\pi_a\alpha_a \le c_p - c_a$ *and* $\pi_s\alpha_s > \frac{1 - \pi_a\alpha_a}{c_a}$*, then* $0 < v_b < v_a < v_p = 1$*, where* $v_b = \frac{c_a}{1 - \pi_a\alpha_a} - \frac{1}{\pi_s\alpha_s}$ *and* $v_a = \frac{c_a}{1 - \pi_a\alpha_a}$ *;*

(VI) If $\pi_a\alpha_a \le c_p - c_a$ *and* $c_a + \pi_a\alpha_a < \pi_s\alpha_s \le \frac{1 - \pi_a\alpha_a}{c_a}$*, then* $0 = v_b < v_a < v_p = 1$*, where* $v_a = \frac{\pi_a\alpha_a + \sqrt{(\pi_a\alpha_a)^2 + 4\pi_s\alpha_s c_a}}{2\pi_s\alpha_s}$ *;*

(VII) If $c_p - c_a < \pi_a\alpha_a < 1 - c_a/c_p$ *and* $\frac{c_p(\pi_a\alpha_a)^2}{(c_p - c_a)^2} < \pi_s\alpha_s \le \frac{1 - \pi_a\alpha_a}{c_a}$*, then* $0 = v_b < v_a < v_p < 1$*, where* $v_a = \frac{\pi_a\alpha_a + \sqrt{(\pi_a\alpha_a)^2 + 4\pi_s\alpha_s c_a}}{2\pi_s\alpha_s}$ *and* $v_p = \frac{c_p - c_a}{\pi_a\alpha_a}$ *.*

Proof of Lemma A.3: We will prove parts (I) and (IV) which are representative of the arguments required. The remaining parts will be omitted due to similarity. For part (I), suppose $\pi_s\alpha_s \le c_p$ is satisfied. This implies $(B, NP) \succ (B, P)$ for all $v \in \mathcal{V}$. Further, $\pi_a\alpha_a > c_p - c_a$ implies $\pi_a\alpha_a + c_a > \pi_s\alpha_s$ which ensures $(B, NP) \succ (B, AP)$ for all $v \in \mathcal{V}$. Because $\pi_s\alpha_s \le c_p < 1$, by (A.10),

$U(v,\sigma)\geq 0$ for all $v\in\mathcal{V}$ if $\sigma(v)=(B,NP)$. Second, $\pi_a\alpha_a\leq c_p-c_a$ and $\pi_s\alpha_s\leq c_a+\pi_a\alpha_a$ together imply the same preferences, and the characterization directly follows.

Next, we consider part (IV). By (A.10), $\sigma(v)=(B,P)$ if and only if $v\geq\max\left(\frac{c_p-c_a}{\pi_a\alpha_a},\frac{c_p}{\pi_s\alpha_s u},c_p\right)$, hence there exists a threshold $v_p\in(0,1]$ such that for all $v\in\mathcal{V}$, $\sigma^*(v)=(B,P)$ if and only if $v\geq v_p$. Moreover, $\sigma(v)\in\{(B,P),(B,NP),(B,AP)\}$ if and only if $v\geq\max\left(c_p,\frac{c_a}{1-\pi_a\alpha_a}\right)$ and $v(1-\pi_s\alpha_s u)\geq 0$. Thus, provided that $u\leq 1/\pi_s\alpha_s$, there exists a $\underline{v}\in(0,1]$ such that a consumer with valuation $v\in\mathcal{V}$ will purchase if and only if $v\geq\underline{v}$. Lastly, $(B,AP)\succ(B,NP)$ if and only if both $u\geq\pi_a\alpha_a/\pi_s\alpha_s$ and $v\geq\frac{c_a}{\pi_s\alpha_s u-\pi_a\alpha_a}$ are satisfied. It follows that if $\pi_a\alpha_a/\pi_s\alpha_s\leq u\leq 1/\pi_s\alpha_s$, then $\underline{v}=v_b\leq v_a\leq v_p$. Together $u\leq 1/\pi_s\alpha_s$ and $\pi_a\alpha_a<1-c_a/c_p$ imply that $v_p=\frac{c_p-c_a}{\pi_a\alpha_a}$. Suppose $u<1/\pi_s\alpha_s$. Then, $v_b=0$, hence $v_a=\frac{\pi_a\alpha_a+\sqrt{(\pi_a\alpha_a)^2+4\pi_s\alpha_s c_a}}{2\pi_s\alpha_s}$. But, $u=v_a\geq 1/\pi_s\alpha_s$ because $\pi_s\alpha_s>\frac{1-\pi_a\alpha_a}{c_a}$ which is a contradiction. Therefore, $u=1/\pi_s\alpha_s$, from which it follows that $v_a=\frac{c_a}{1-\pi_a\alpha_a}$ and $v_b=\frac{c_a}{1-\pi_a\alpha_a}-\frac{1}{\pi_s\alpha_s}$. $\square$

Lemma A.4 *The consumer market equilibrium for an open-source software product under a tax τ is given by the following: If $\tau\leq c_a$, then*

(I) If $c_p+\tau-c_a<\pi_a\alpha_a<1-\frac{c_a}{c_p+\tau}$ and $(c_p+\tau-c_a)^2(c_p+\tau-c_a-c_p\pi_a\alpha_a-\tau\pi_a\alpha_a)\pi_s\alpha_s>c_p(\pi_a\alpha_a)^2(c_p+\tau-c_a-c_p\pi_a\alpha_a)$, then $0<v_b<v_a<v_p<1$, where $v_p=\frac{c_p+\tau-c_a}{\pi_a\alpha_a}$, $v_a=\frac{c_a-\tau}{\pi_s\alpha_s u-\pi_a\alpha_a}$, and $v_b=\frac{\tau}{1-\pi_s\alpha_s u}$. If $\tau<c_a$, $u\in(\frac{\pi_a\alpha_a}{\pi_s\alpha_s},\frac{1}{\pi_s\alpha_s})$ satisfies $u(\pi_s\alpha_s u-\pi_a\alpha_a)(\pi_s\alpha_s u-1)=c_a\pi_s\alpha_s u-(c_a-\tau+\tau\pi_a\alpha_a)$. If $\tau=c_a$, $u=\frac{\pi_a\alpha_a}{\pi_s\alpha_s}$;

(II) If $\pi_a\alpha_a\leq c_p+\tau-c_a$ and $(1-c_a-\pi_a\alpha_a)\pi_s\alpha_s>(\pi_a\alpha_a+c_a-\tau)(1-c_a-\pi_a\alpha_a+\tau)$, then $0<v_b<v_a<1$, where $v_a=\frac{c_a-\tau}{\pi_s\alpha_s u-\pi_a\alpha_a}$ and $v_b=\frac{\tau}{1-\pi_s\alpha_s u}$. If $\tau<c_a$, $u\in(\frac{\pi_a\alpha_a}{\pi_s\alpha_s},\frac{1}{\pi_s\alpha_s})$ satisfies $u(\pi_s\alpha_s u-\pi_a\alpha_a)(\pi_s\alpha_s u-1)=c_a\pi_s\alpha_s u-(c_a-\tau+\tau\pi_a\alpha_a)$. If $\tau=c_a$, $u=\frac{\pi_a\alpha_a}{\pi_s\alpha_s}$;

(III) If either (i) $\pi_a\alpha_a\geq 1-\frac{c_a}{c_p+\tau}$ and $(1-c_p-\tau)\pi_s\alpha_s>c_p(1-c_p)$, or (ii) $c_p+\tau-c_a<\pi_a\alpha_a<1-\frac{c_a}{c_p+\tau}$, $c_p(1-c_p)<(1-c_p-\tau)\pi_s\alpha_s$, and $(c_p+\tau-c_a)^2(c_p+\tau-c_a-c_p\pi_a\alpha_a-\tau\pi_a\alpha_a)\pi_s\alpha_s\leq c_p(\pi_a\alpha_a)^2(c_p+\tau-c_a-c_p\pi_a\alpha_a)$, then $0<v_b<v_p<1$, where $v_p=\frac{c_p}{\pi_s\alpha_s u}$, $v_b=\frac{\tau}{1-\pi_s\alpha_s u}$, and $u\in(0,\frac{1}{\pi_s\alpha_s})$ satisfies $\pi_s\alpha_s u^2(\pi_s\alpha_s u-1)=(c_p+\tau)\pi_s\alpha_s u-c_p$;

(IV) If either (i) $\pi_a\alpha_a\leq c_p+\tau-c_a$ and $(1-\pi_a\alpha_a-c_a)\pi_s\alpha_s\leq(1+\tau-\pi_a\alpha_a-c_a)(\pi_a\alpha_a+c_a-\tau)$, or (ii) $\pi_a\alpha_a>c_p+\tau-c_a$ and $(1-c_p-\tau)\pi_s\alpha_s\leq c_p(1-c_p)$, then $0<v_b<1$, where $v_b=\frac{\tau}{1-\pi_s\alpha_s u}$ and $u=\frac{1+\pi_s\alpha_s-\sqrt{(1+\pi_s\alpha_s)^2-4\pi_s\alpha_s(1-\tau)}}{2\pi_s\alpha_s}$.

If $\tau>c_a$, then

(I) If either (i) $c_p+\tau-c_a<\pi_a\alpha_a\leq(c_ac_p+\tau-c_a)/\tau$ and $(\pi_a\alpha_a-c_p-\tau+c_a)\pi_s\alpha_s>c_p(\pi_a\alpha_a-c_p)$, or (ii) $1-c_a(1-c_p)/\tau<\pi_a\alpha_a<1-c_a/(c_p+\tau)$ and $c_a^2((c_p+\tau)(1-\pi_a\alpha_a)-c_a)\pi_s\alpha_s>(c_a-\tau(1-\pi_a\alpha_a))^2(1-\pi_a\alpha_a)$, then $0<v_a<v_b<v_p<1$, where $v_a=\frac{c_a}{1-\pi_a\alpha_a}$, $v_b=\frac{\tau-c_a}{\pi_a\alpha_a-\pi_s\alpha_s u}$, and $v_p=\frac{c_p}{\pi_s\alpha_s u}$. $u\in(0,\frac{\pi_a\alpha_a}{\pi_s\alpha_s})$ satisfies $\pi_s\alpha_s u^2(\pi_s\alpha_s u-\pi_a\alpha_a)=\pi_s\alpha_s(c_p+\tau-c_a)u-c_p\pi_a\alpha_a$;

(II) If either (i) $\tau-c_a<\pi_a\alpha_a\leq c_p+\tau-c_a$ and $(1-\pi_a\alpha_a-c_a)c_a\pi_s\alpha_s>(c_a-\tau+\tau\pi_a\alpha_a)(1-\pi_a\alpha_a)$, or (ii) $c_p+\tau-c_a<\pi_a\alpha_a<1-\frac{c_a(1-c_p)}{\tau}$, $(c_a-\tau+\tau\pi_a\alpha_a)(1-\pi_a\alpha_a)<(1-\pi_a\alpha_a-c_a)c_a\pi_s\alpha_s$, and $(\pi_a\alpha_a-c_p-\tau+c_a)\pi_s\alpha_s\leq c_p(\pi_a\alpha_a-c_p)$, then $0<v_a<v_b<1$, where $v_a=c_a/(1-\pi_a\alpha_a)$, $v_b=\frac{\tau-c_a}{\pi_a\alpha_a-\pi_s\alpha_s u}$, and $u=\frac{\pi_a\alpha_a+\pi_s\alpha_s-\sqrt{(\pi_a\alpha_a+\pi_s\alpha_s)^2-4\pi_s\alpha_s(\pi_a\alpha_a-\tau+c_a)}}{2\pi_s\alpha_s}$;

(III) If either (i) $1-\frac{c_a(1-c_p)}{\tau}<\pi_a\alpha_a<1-\frac{c_a}{c_p+\tau}$, $c_p(1-c_p)<(1-c_p-\tau)\pi_s\alpha_s$, *and* $c_a^2((c_p+\tau)(1-\pi_a\alpha_a)-c_a)\pi_s\alpha_s \leq (\tau\pi_a\alpha_a-\tau+c_a)^2(1-\pi_a\alpha_a)$, *or (ii)* $\pi_a\alpha_a \geq 1-\frac{c_a}{c_p+\tau}$ *and* $\pi_s\alpha_s(1-c_p-\tau)>c_p(1-c_p)$, *then* $0<v_b<v_p<1$, *where* $v_b=\frac{\tau}{1-\pi_s\alpha_s u}$ *and* $v_p=\frac{c_p}{\pi_s\alpha_s u}$. $u\in(0,\frac{1}{\pi_s\alpha_s})$ *solves* $\pi_s\alpha_s u^2(\pi_s\alpha_s u-1)=(c_p+\tau)\pi_s\alpha_s u-c_p$ *;*

(IV) If either (i) $1-c_a/\tau<\pi_a\alpha_a\leq 1-\frac{c_a(1-c_p)}{\tau}$ *and* $\pi_s\alpha_s c_a(1-\pi_a\alpha_a-c_a)\leq(1-\pi_a\alpha_a)(c_a-\tau+\tau\pi_a\alpha_a)$, *or (ii)* $\pi_a\alpha_a>1-\frac{c_a(1-c_p)}{\tau}$ *and* $\pi_s\alpha_s(1-\tau-c_p)\leq c_p(1-c_p)$, *then* $0<v_b<1$, *where* $v_b=1-u$ *and* $u=\frac{1+\pi_s\alpha_s-\sqrt{(1+\pi_s\alpha_s)^2-4\pi_s\alpha_s(1-\tau)}}{2\pi_s\alpha_s}$ *;*

(V) If $\pi_a\alpha_a\leq\tau-c_a$, *then* $0<v_a<1$, *where* $v_a=\frac{c_a}{1-\pi_a\alpha_a}$ *and* $u=0$.

Note that the consumer market equilibrium structures of (III) and (IV) when $\tau\leq c_a$ *are the same as those when* $\tau>c_a$.

Proof of Lemma A.4: The results can be derived as a special case of Lemma A.2 by setting $p=\tau$ and $\delta=0$. One thing notable is that the consumer market equilibrium structures may change depending on whether $\tau\leq c_a$ or not. □

Proof of Lemma 4: This follows from taking $\pi_s\alpha_s\to\infty$ in Lemma A.3. □

Proof of Lemma 5: Technically, we prove that there exists an α such that for $\alpha_s>\alpha$, τ^* is set so that

(i) if $c_p-\pi_a\alpha_a<c_a<c_p(1-\pi_a\alpha_a)$, then σ^* is characterized by $0<v_b<v_a<v_p<1$ under the optimal tax.

(ii) if $\frac{(\pi_a\alpha_a)^2}{1-\pi_a\alpha_a}<c_a\leq c_p-\pi_a\alpha_a$, then σ^* is characterized by $0<v_b<v_a<1$ under the optimal tax.

(iii) if $c_a\leq c_p-\pi_a\alpha_a$ and $c_a\leq\frac{(\pi_a\alpha_a)^2}{1-\pi_a\alpha_a}$, then σ^* is characterized by $0<v_a<v_b<1$ under the optimal tax.

Suppose that $0<v_b<1$. By Lemma A.4, we obtain $v_b=\frac{-1+\pi_s\alpha_s+\sqrt{(1-\pi_s\alpha_s)^2+4\tau\pi_s\alpha_s}}{2\pi_s\alpha_s}$. Substituting this into the social welfare function $W_N(\tau)=\int_{v_b(\tau)}^1 v-(1-v_b(\tau))\pi_s\alpha_s v dv$ for this case, we find that the social welfare function is then given by

$$W_N(\tau)=\frac{\tau(1-\tau)}{\pi_s\alpha_s}-\frac{\tau(1-\tau)(1-3\tau)}{2(\pi_s\alpha_s)^2}+O\left(\frac{1}{(\pi_s\alpha_s)^3}\right). \tag{A.66}$$

The interior maximizing price is given by

$$\tau_N=\frac{1}{2}-\frac{3}{16\pi_s\alpha_s}-\frac{3}{64(\pi_s\alpha_s)^2}+O\left(\frac{1}{(\pi_s\alpha_s)^3}\right). \tag{A.67}$$

By substituting (A.67) into (A.66), we have that the social welfare in this case, when the solution is interior, is given by

$$W_N(\tau_N)=\frac{1}{4\pi_s\alpha_s}-\frac{1}{16(\pi_s\alpha_s)^2}+O\left(\frac{1}{(\pi_s\alpha_s)^3}\right). \tag{A.68}$$

On the other hand, suppose that $0<v_a<v_b<v_p<1$ is induced. By Lemma A.4, we obtain $v_a = \frac{c_a}{1-\pi_a\alpha_a}$, $v_b = \frac{c_p-c_a+\tau}{\pi_a\alpha_a} - \frac{c_p^2\pi_a\alpha_a}{(c_p-c_a+\tau)^2\pi_s\alpha_s} + \frac{2c_p^3(c_a-\tau)(\pi_a\alpha_a)^3}{(c_a-c_p-\tau)^5(\pi_s\alpha_s)^2} + O\left(\frac{1}{(\pi_s\alpha_s)^3}\right)$, and $v_p = \frac{c_p-c_a+\tau}{\pi_a\alpha_a} + \frac{c_p(\tau-c_a)\pi_a\alpha_a}{(c_p-c_a+\tau)^2\pi_s\alpha_s} - \frac{c_p^2(\tau-c_a)(c_p+c_a-\tau)(\pi_a\alpha_a)^3}{(c_p-c_a+\tau)^5(\pi_s\alpha_s)^2} + O\left(\frac{1}{(\pi_s\alpha_s)^3}\right)$.

Substituting this into the social welfare function

$$W_A(\tau) = \int_{v_a(\tau)}^1 vdv - \left(\int_{v_a(\tau)}^{v_b(\tau)} c_a + \pi_a\alpha_a vdv + \int_{v_b(\tau)}^{v_p(\tau)} (v_p(\tau)-v_b(\tau))\pi_s\alpha_s vdv + c_p(1-v_p(\tau))\right),$$

we find that the social welfare function is then given by

$$W_A(\tau) = \frac{1}{2}\left(1-2c_p+\frac{(c_p-c_a)^2-\tau^2}{\pi_a\alpha_a}+\frac{c_a^2}{1-\pi_a\alpha_a}\right)+\frac{c_p^2\tau\pi_a\alpha_a}{(c_p-c_a+\tau)^2\pi_s\alpha_s}- \frac{c_p^3(c_p-c_a-3\tau)(c_a-\tau)(\pi_a\alpha_a)^3}{2(c_p-c_a+\tau)^5(\pi_s\alpha_s)^2}+O\left(\frac{1}{(\pi_s\alpha_s)^3}\right). \quad (A.69)$$

The interior maximizing price is given by

$$\tau_A = \frac{(c_p\pi_a\alpha_a)^2}{(c_p-c_a)^2\pi_s\alpha_s} - \frac{7c_p^3(\pi_a\alpha_a)^4}{2(c_p-c_a)^4(\pi_s\alpha_s)^2} + O\left(\frac{1}{(\pi_s\alpha_s)^3}\right). \quad (A.70)$$

By substituting (A.70) into (A.69), we have that the social welfare in this case, when the solution is interior, is given by

$$W_A(\tau_A) = \frac{1}{2}\left(1-2c_p+\frac{(c_p-c_a)^2}{\pi_a\alpha_a}+\frac{c_a^2}{1-\pi_a\alpha_a}\right)+\frac{(c_p\pi_a\alpha_a)^3}{2(c_p-c_a)^3(\pi_s\alpha_s)^2}+O\left(\frac{1}{(\pi_s\alpha_s)^3}\right). \quad (A.71)$$

Next, suppose that $0<v_b<v_a<v_p<1$ is induced. By Lemma A.4, we obtain $v_a = \frac{c_a}{1-\pi_a\alpha_a} + \frac{\tau(c_a-\tau(1-\pi_a\alpha_a))}{c_a^2\pi_s\alpha_s} + \frac{\tau(c_a-\tau)(1-\pi_a\alpha_a)(c_a-\tau(1-\pi_a\alpha_a))(c_a-2\tau(1-\pi_a\alpha_a))}{c_a^5(\pi_s\alpha_s)^2} + O\left(\frac{1}{(\pi_s\alpha_s)^3}\right)$, $v_b = \frac{c_a}{1-\pi_a\alpha_a} - \frac{(c_a-\tau)(c_a-(1-\pi_a\alpha_a)\tau}{c_a^2\pi_s\alpha_s} - \frac{\tau(c_a-\tau)(1-\pi_a\alpha_a)(-2c_a+2\tau+(c_a-2\tau)\pi_a\alpha_a)(c_a-\tau(1-\pi_a\alpha_a))}{c_a^5(\pi_s\alpha_s)^2} + O\left(\frac{1}{(\pi_s\alpha_s)^3}\right)$, and $v_p = \frac{c_p-(c_a-\tau)}{\pi_a\alpha_a}$. Again, substituting this into the social welfare function, which for this case is given by $W(\tau) = \int_{v_b(\tau)}^1 vdv - \left(\int_{v_b(\tau)}^{v_a(\tau)} (v_a(\tau)-v_b(\tau\right.$

we find that the social welfare function is then given by

$$W_B(\tau) = \frac{1}{2}\left(1-2c_p+\frac{(c_p-c_a)^2-\tau^2}{\pi_a\alpha_a}+\frac{c_a^2}{1-\pi_a\alpha_a}\right)+ \frac{\tau(c_a-\tau(1-\pi_a\alpha_a))}{c_a\pi_s\alpha_s}+\frac{(c_a-\tau)\tau(1-\pi_a\alpha_a)(c_a-\tau(1-\pi_a\alpha_a))(c_a-3\tau(1-\pi_a\alpha_a))}{2c_a^4(\pi_s\alpha_s)^2}+O\left(\frac{1}{(\pi_s\alpha_s)^2}\right). \quad (A.72)$$

The interior maximizing price is given by

$$\tau_B = \frac{\pi_a\alpha_a}{\pi_s\alpha_s} + \frac{\pi_a\alpha_a(1-\pi_a\alpha_a)(1-4\pi_a\alpha_a)}{2c_a(\pi_s\alpha_s)^2} + O\left(\frac{1}{(\pi_s\alpha_s)^3}\right). \quad (A.73)$$

By substituting (A.73) into (A.72), we have that the social welfare in this case, when the solution is interior, is given by

$$W_B(\tau_B) = \frac{1}{2}\left(1 - 2c_p + \frac{(c_p - c_a)^2}{\pi_a\alpha_a} + \frac{c_a^2}{1 - \pi_a\alpha_a}\right) + \frac{\pi_a\alpha_a}{2(\pi_s\alpha_s)^2} + O\left(\frac{1}{(\pi_s\alpha_s)^3}\right). \tag{A.74}$$

Next, suppose that $0 < v_b < v_p < 1$ is induced. By Lemma A.4, we obtain $v_b = c_p + \tau - \frac{c_p^2}{(c_p+\tau)^2\pi_s\alpha_s} + O\left(\frac{1}{(\pi_s\alpha_s)^2}\right)$ and $v_p = c_p + \tau + \frac{\tau c_p}{(c_p+\tau)^2\pi_s\alpha_s} + O\left(\frac{1}{(\pi_s\alpha_s)^2}\right)$. Again, substituting this into the social welfare function, which for this case is given by

$$W_P(\tau) = \int_{v_b(\tau)}^{1} v dv - \left(\int_{v_b(\tau)}^{v_p(\tau)} (v_p(\tau) - v_b(\tau))\pi_s\alpha_s v dv + c_p(1 - v_p(\tau))\right)$$

, we find that the social welfare function is then given by

$$W_P(\tau) = \frac{1}{2}\left((1 - c_p)^2 - \tau^2\right) + \frac{c_p^2\tau}{(c_p + \tau)^2\pi_s\alpha_s} + O\left(\frac{1}{(\pi_s\alpha_s)^2}\right). \tag{A.75}$$

The interior maximizing price is given by

$$\tau_P = \frac{1}{\pi_s\alpha_s} + O\left(\frac{1}{(\pi_s\alpha_s)^2}\right). \tag{A.76}$$

By substituting (A.76) into (A.75), we have that the social welfare in this case, when the solution is interior, is given by

$$W_P(\tau_P) = \frac{1}{2}(1 - c_p)^2 + O\left(\frac{1}{(\pi_s\alpha_s)^2}\right). \tag{A.77}$$

Next, suppose that $0 < v_a < 1$ is induced in equilibrium. By Lemma A.4, we obtain $v_a = \frac{c_a}{1-\pi_a\alpha_a}$. Again, substituting this into the social welfare function, which for this case is given by $W(\tau) = \int_{v_a(\tau)}^{1} v dv - \int_{v_a(\tau)}^{1} c_a + \pi_a\alpha_a v dv$, we find that the optimal social welfare function is then

$$W_a(\tau) = \frac{(1 - c_a - \pi_a\alpha_a)^2}{2(1 - \pi_a\alpha_a)}, \tag{A.78}$$

which doesn't depend on τ.

Next, suppose that $0 < v_a < v_b < 1$ is induced in equilibrium. By Lemma A.4, we obtain $v_a = \frac{c_a}{1-\pi_a\alpha_a}$ and $v_b = \frac{\pi_s\alpha_s - \pi_a\alpha_a + \sqrt{(\pi_s\alpha_s - \pi_a\alpha_a)^2 + 4\pi_s\alpha_s(\tau - c_a)}}{2\pi_s\alpha_s}$. Again, substituting this into the social welfare function, which for this case is given by

$$W_{ab}(\tau) = \int_{v_a(\tau)}^{1} v dv - \left(\int_{v_a(\tau)}^{v_b(\tau)} c_a + \pi_a\alpha_a v dv + \int_{v_b(\tau)}^{1} (1 - v_b(\tau))\pi_s\alpha_s v dv\right),$$

we find that the optimal social welfare function is then

$$W_{ab}(\tau) = \frac{(1-c_a-\pi_a\alpha_a)^2}{2(1-\pi_a\alpha_a)} + \frac{\tau(c_a-\tau+\pi_a\alpha_a)}{\pi_s\alpha_s} - \frac{(c_a-\tau)(c_a-3\tau+\pi_a\alpha_a)(c_a-\tau+\pi_a\alpha_a)}{2(\pi_s\alpha_s)^2} + O\left(\frac{1}{(\pi_s\alpha_s)^3}\right). \tag{A.79}$$

The interior maximizing price is given by

$$\tau_{ab} = \frac{1}{2}(c_a+\pi_a\alpha_a) + \frac{(c_a-3\pi_a\alpha_a)(c_a+\pi_a\alpha_a)}{16\pi_s\alpha_s} + \frac{(c_a-\pi_a\alpha_a)(3c_a-\pi_a\alpha_a)(c_a+\pi_a\alpha_a)}{64(\pi_s\alpha_s)^2} + O\left(\frac{1}{(\pi_s\alpha_s)^3}\right). \tag{A.80}$$

By substituting (A.80) into (A.79), we have that the social welfare in this case, when the solution is interior, is given by

$$W_{ab}(\tau_{ab}) = \frac{(1-c_a-\pi_a\alpha_a)^2}{2(1-\pi_a\alpha_a)} + \frac{(c_a+\pi_a\alpha_a)^2}{4\pi_s\alpha_s} + \frac{(c_a-\pi_a\alpha_a)(c_a+\pi_a\alpha_a)^2}{16(\pi_s\alpha_s)^2} + O\left(\frac{1}{(\pi_s\alpha_s)^3}\right). \tag{A.81}$$

Finally, suppose that $0 < v_b < v_a < 1$ is induced in equilibrium. By Lemma A.4, we obtain $v_a = \frac{c_a}{1-\pi_a\alpha_a} + \frac{\tau(c_a-\tau(1-\pi_a\alpha_a))}{c_a^2\pi_s\alpha_s} + \frac{\tau(c_a-\tau)(1-\pi_a\alpha_a)(c_a-\tau(1-\pi_a\alpha_a))(c_a-2\tau(1-\pi_a\alpha_a))}{c_a^5(\pi_s\alpha_s)^2} + O\left(\frac{1}{(\pi_s\alpha_s)^3}\right)$ and $v_b = \frac{c_a}{1-\pi_a\alpha_a} - \frac{(c_a-\tau)(c_a-\tau(1-\pi_a\alpha_a))}{c_a^2\pi_s\alpha_s} - \frac{\tau(c_a-\tau)(1-\pi_a\alpha_a)(-2c_a+2\tau+(c_a-2\tau)\pi_a\alpha_a)(c_a-\tau(1-\pi_a\alpha_a))}{c_a^5(\pi_s\alpha_s)^2} + O\left(\frac{1}{(\pi_s\alpha_s)^3}\right)$. Again, substituting this into the social welfare function, which for this case is given by

$$W_{ba}(\tau) = \int_{v_b(\tau)}^1 v dv - \left(\int_{v_b(\tau)}^{v_a(\tau)} (v_a(\tau)-v_b(\tau))\pi_s\alpha_s v dv + \int_{v_a(\tau)}^1 c_a + \pi_a\alpha_a v dv\right),$$

we find that the optimal social welfare function is then

$$W_{ba}(\tau) = \frac{(1-c_a-\pi_a\alpha_a)^2}{2(1-\pi_a\alpha_a)} + \frac{\tau(c_a-\tau(1-\pi_a\alpha_a))}{c_a\pi_s\alpha_s} + \frac{\tau(c_a-\tau)(1-\pi_a\alpha_a)(c_a-\tau(1-\pi_a\alpha_a))(c_a-3\tau(1-\pi_a\alpha_a))}{2c_a^4(\pi_s\alpha_s)^2} + O\left(\frac{1}{(\pi_s\alpha_s)^2}\right). \tag{A.82}$$

The interior maximizing price is given by

$$\tau_{ba} = \frac{1}{2}\left(\frac{c_a}{1-\pi_a\alpha_a}\right) - \frac{1-3\pi_a\alpha_a}{16(1-\pi_a\alpha_a)\pi_s\alpha_s} - \frac{(2-\pi_a\alpha_a)(1-3\pi_a\alpha_a)}{(64c_a(1-\pi_a\alpha_a)(\pi_s\alpha_s)^2} + O\left(\frac{1}{(\pi_s\alpha_s)^3}\right). \tag{A.83}$$

By substituting (A.83) into (A.82), we have that the social welfare in this case, when the solution is interior, is given by

$$W_{ba}(\tau_{ba}) = \frac{(1-c_a-\pi_a\alpha_a)^2}{2(1-\pi_a\alpha_a)} + \frac{c_a}{4(1-\pi_a\alpha_a)\pi_s\alpha_s} - \frac{(1-2\pi_a\alpha_a}{32(1-\pi_a\alpha_a)(\pi_s\alpha_s)^2} + O\left(\frac{1}{(\pi_s\alpha_s)^3}\right). \tag{A.84}$$

To prove Lemma 5, we proceed as follows. We first focus on the case when $0 < v_b < v_a < v_p < 1$ is induced in equilibrium. We find the conditions under which the interior optimal price for this case indeed induces the conjectured market structure. Further, we show that the induced welfare is greater than the maximal welfare of the other cases (under their respective optimal taxes) when the conditions specified in the lemma are met. We then proceed to the remaining cases of $0 < v_b < v_a < 1$

and $0 < v_a < v_b < 1$ using the same steps.

Suppose that $0 < v_b < v_a < v_p < 1$ is induced in equilibrium. For the optimal tax of this case, in (A.73), to induce this case, we need to have that (A.73) satisfies (I) of Lemma A.4 (when $\tau \leq c_a$) for sufficient high $\pi_s \alpha_s$. Omitting the algebra, the condition under which the optimal tax of $0 < v_b < v_a < v_p < 1$, given in (A.73), indeed induces the market structure $0 < v_b < v_a < v_p < 1$ is given by

$$c_p - \pi_a \alpha_a < c_a < c_p(1 - \pi_a \alpha_a).$$

Next, we show that under the above conditions, the interior optimal welfare of $0 < v_b < v_a < v_p < 1$ dominates the interior optimal welfares (and possible boundary extrema) in all the other cases. Just by comparing the expressions, we see that the interior optimal welfare of $0 < v_b < v_a < v_p < 1$ dominates the interior optimal welfares of all the cases except for possibly $0 < v_a < v_b < v_p < 1$. Specifically, (A.74) is greater than (A.84), (A.81), (A.78), (A.77), and (A.68). To complete the proof for this case, we next show that the interior solution for $0 < v_a < v_b < v_p < 1$ can't induce that market structure. To show that the interior solution for $0 < v_a < v_b < v_p < 1$ doesn't induce $0 < v_a < v_b < v_p < 1$, note that from (I) of Lemma A.4 (when $\tau > c_a$), one of the conditions is $\tau > c_a$. However, looking at (A.70), we see that for sufficiently high $\pi_s \alpha_s$, we'll have that $\tau_A < c_a$. Therefore, the interior solution for $0 < v_a < v_b < v_p < 1$ can't induce that market structure. In other words, if, given the parameters, $0 < v_a < v_b < v_p < 1$ can be induced by some τ, then the maximal welfare of $0 < v_a < v_b < v_p < 1$ occurs at a boundary point for that market structure. It follows that (A.74) dominates the boundary between $0 < v_a < v_b < v_p < 1$ and any other market structures, since it dominates the interior optimal welfares in the other structures. This proves part (i) of Lemma 5.

Next, suppose that $0 < v_b < v_a < 1$ is induced in equilibrium. For the optimal tax of this case, in (A.83), to induce this case, we need to have that (A.83) satisfies (II) of Lemma A.4 (when $\tau \leq c_a$) for sufficient high $\pi_s \alpha_s$. Omitting the algebra, the condition under which the optimal tax of $0 < v_b < v_a < 1$, given in (A.83), indeed induces the market structure $0 < v_b < v_a < 1$ is given by $\pi_a \alpha_a < \frac{1}{2}$ and $c_a \leq \frac{2(c_p - \pi_a \alpha_a)(1 - \pi_a \alpha_a)}{1 - 2\pi_a \alpha_a}$.

We show next that under the conditions specified in the lemma for this case, the interior optimal welfare of $0 < v_b < v_a < 1$ dominates the interior optimal welfares (and possible boundary extrema) in all the other cases. Just by comparing the expressions, we see that the interior optimal welfare of $0 < v_b < v_a < 1$, given in (A.84), always dominates the interior optimal welfares of cases $0 < v_a < 1$ and $0 - v_b - 1$, given in (A.78) and (A.68) respectively. Moreover, when $c_a \leq c_p - \pi_a \alpha_a$, we'll have that the interior optimal welfare of $0 < v_b < v_a < 1$ is greater than the interior optimal welfare of $0 - v_b - v_p - 1$, given in (A.77). We'll now show that under the conditions specified for this case, namely $\frac{(\pi_a \alpha_a)^2}{1 - \pi_a \alpha_a} < c_a \leq c_p - \pi_a \alpha_a$, it will be the case that there's no τ which can induce either $0 < v_b < v_a < v_p < 1$ or $0 < v_a < v_b < v_p < 1$. Specifically, using part (I) of Lemma A.4 for when $\tau \leq c_a$, for the case $0 < v_b < v_a < v_p < 1$ to be induced for some τ when $\pi_s \alpha_s$ gets sufficiently big, we need $(1 + c_p)(1 - \pi_a \alpha_a) > c_a > c_p - \pi_a \alpha_a$ to hold. Similarly, using part (I) of Lemma A.4 for when $\tau > c_a$, for the case $0 < v_b < v_a < v_p < 1$ to be induced for some τ when $\pi_s \alpha_s$ gets sufficiently big, we also need $c_a > c_p - \pi_a \alpha_a$. Therefore, when $c_a \leq c_p - \pi_a \alpha_a$ (as specified in the lemma for this case), it'll be the case that no τ can induce either $0 < v_b < v_a < v_p < 1$ or $0 < v_a < v_b < v_p < 1$ in equilibrium. Lastly, we need to compare the interior optimal welfare of $0 < v_b < v_a < 1$ against the interior optimal welfare of $0 < v_a < v_b < 1$. Note that the conditions for the interior optimal tax of $0 < v_a < v_b < 1$ to indeed induce $0 < v_a < v_b < 1$ are given by $2c_p \geq c_a + \pi_a \alpha_a$ and $c_a < \pi_a \alpha_a$. By comparing (A.84) and

(A.81), we see that the interior optimal welfare of $0<v_b<v_a<1$ dominates the interior optimal welfare of $0<v_a<v_b<1$ when $c_a \geq \frac{(\pi_a\alpha_a)^2}{1-\pi_a\alpha_a}$. Since $\frac{(\pi_a\alpha_a)^2}{1-\pi_a\alpha_a} < \pi_a\alpha_a$ and $c_a + \pi_a\alpha_a > \frac{1}{2}(c_a + \pi_a\alpha_a)$ for $\pi_a\alpha_a < \frac{1}{2}$, it follows that we need to have $c_a \geq \frac{(\pi_a\alpha_a)^2}{1-\pi_a\alpha_a}$ as a condition for $0<v_b<v_a<1$ to be induced in equilibrium. We note that $\pi_a\alpha_a < \frac{1}{2}$ must hold in order for $\frac{(\pi_a\alpha_a)^2}{1-\pi_a\alpha_a} < c_a \leq c_p - \pi_a\alpha_a$ to hold for some c_a and c_p, and $\frac{(\pi_a\alpha_a)^2}{1-\pi_a\alpha_a} < c_a \leq c_p - \pi_a\alpha_a$ is a subset of $c_a \leq \frac{2(c_p-\pi_a\alpha_a)(1-\pi_a\alpha_a)}{1-2\pi_a\alpha_a}$, so that when the conditions of the lemma hold, $0<v_b<v_a<1$ will be induced in equilibrium. This completes the proof for this case.

Lastly, suppose that $0<v_a<v_b<1$ is induced in equilibrium. For the optimal tax of this case, in (A.80), to induce this case, we need to have that (A.80) satisfies (II) of Lemma A.4 (when $\tau > c_a$) for sufficient high $\pi_s\alpha_s$. Omitting the algebra, the conditions under which the optimal tax of $0<v_a<v_b<1$, given in (A.80), indeed induces the market structure $0<v_a<v_b<1$ are given by $c_a < \pi_a\alpha_a$ and $c_a \leq 2c_p - \pi_a\alpha_a$.

We show next that under the conditions specified in the lemma for this case, the interior optimal welfare of $0<v_a<v_b<1$ dominates the interior optimal welfares (and possible boundary extrema) in all the other cases. In the same way as the previous case, we have that the interior optimal welfare of $0<v_a<v_b<1$, given in (A.81), always dominates the interior optimal welfares of cases $0<v_a<1$ and $0<v_b<1$, given in (A.78) and (A.68) respectively. Moreover, when $c_a \leq c_p - \pi_a\alpha_a$, we'll have that the interior optimal welfare of $0<v_a<v_b<1$ is greater than the interior optimal welfare of $0-v_b-v_p-1$, given in (A.77). For the same reason as in the previous case of $0<v_b<v_a<1$, we have that there's no τ which can induce either $0<v_b<v_a<v_p<1$ or $0<v_a<v_b<v_p<1$. Therefore, when $c_a \leq c_p - \pi_a\alpha_a$ (as specified in the lemma for this case), it'll be the case that no τ can induce either $0<v_b<v_a<v_p<1$ or $0<v_a<v_b<v_p<1$ in equilibrium. Lastly, we need to compare the interior optimal welfare of $0<v_a<v_b<1$ against the interior optimal welfare of $0<v_b<v_a<1$. Again, from the previous case, we have that the condition is $c_a < \frac{(\pi_a\alpha_a)^2}{1-\pi_a\alpha_a}$. Note that when $\pi_a\alpha_a \leq \frac{1}{2}$, we have $\frac{(\pi_a\alpha_a)^2}{1-\pi_a\alpha_a} < \pi_a\alpha_a$ and when $\pi_a\alpha_a > \frac{1}{2}$, we have $c_p - \pi_a\alpha_a < \pi_a\alpha_a$. Therefore, $c_a < \frac{(\pi_a\alpha_a)^2}{1-\pi_a\alpha_a}$ and $c_a \leq c_p - \pi_a\alpha_a$ imply $c_a < \pi_a\alpha_a$. Also, $c_a \leq c_p - \pi_a\alpha_a$ implies $c_a \leq 2c_p - \pi_a\alpha_a$. From the above, we have that the conditions $c_a < \frac{(\pi_a\alpha_a)^2}{1-\pi_a\alpha_a}$ and $c_a \leq c_p - \pi_a\alpha_a$ imply that the optimal tax of $0<v_a<v_b<1$ indeed induces that market structure in equilibrium. This completes the proof for this case and concludes the lemma. □

5 Proofs of Propositions

Proof of Proposition 5: From Lemma A.3, it becomes clear that the absence of an automated patching option is a special case characterized by:

(A) If $\pi_s\alpha_s \leq c_p$, then $0=v_b<v_p=1$;

(B) If $c_p<\pi_s\alpha_s \leq 1/c_p$, then $0=v_b<v_p<1$, where $v_p=\sqrt{\frac{c_p}{\pi_s\alpha_s}}$;

(C) If $\pi_s\alpha_s>1/c_p$, then $0<v_b<v_p<1$, where $v_b=c_p-1/\pi_s\alpha_s$ and $v_p=c_p$.

For part (i) of the proposition, suppose that $c_p<\pi_s\alpha_s \leq \frac{c_p(\pi_a\alpha_a)^2}{(c_p-c_a)^2}$. By part (II) of Lemma A.3, $u=\sqrt{\frac{c_p}{\pi_s\alpha_s}}$. Part (B) above is also satisfied since $\frac{c_p(\pi_a\alpha_a)^2}{(c_p-c_a)^2}<1/c_p$, and hence $\tilde{u}=\sqrt{\frac{c_p}{\pi_s\alpha_s}}$. Suppose $\pi_s\alpha_s \leq c_p$. By part (I) of Lemma A.3 and part (A) above, $u=\tilde{u}=1$. For part (ii) of the proposition, by part (VII) of Lemma A.3, $u=\frac{\pi_a\alpha_a+\sqrt{(\pi_a\alpha_a)^2+4\pi_s\alpha_s c_a}}{2\pi_s\alpha_s}$. Because $c_p<\frac{c_p(\pi_a\alpha_a)^2}{(c_p-c_a)^2}<\frac{1}{c_p}<\frac{1-\pi_a\alpha_a}{c_a}$, $\tilde{u}$ satisfies either $\tilde{u}=\sqrt{\frac{c_p}{\pi_s\alpha_s}}$ or $\tilde{u}=1/\pi_s\alpha_s$. In either case, $u<\tilde{u}$. For part (iii) of the proposition, by part (IV) of Lemma A.3 and part (C) above, $u=\tilde{u}=1/\pi_s\alpha_s$, which completes the proof. ■

Proof of Proposition 6: By Lemma 4 and Lemma 5, for sufficiently high $\pi_s\alpha_s$, we have the market structure $0<v_b<v_a<v_p<1$ being induced both under the status quo setting without the tax as well as under the optimal tax when $c_p-\pi_a\alpha_a<c_a<c_p(1-\pi_a\alpha_a)$. The social welfare function of this case is given by (A.72). The welfare under the status quo case of no tax is given as

$$W_{B,\text{ Status Quo}}=\frac{1}{2}\left(1-2c_p+\frac{(c_a-c_p)^2}{\pi_a\alpha_a}+\frac{c_a^2}{1-\pi_a\alpha_a}\right)+O\left(\frac{1}{(\pi_s\alpha_s)}\right),$$

which is the limit of (A.72) as $\tau\to 0$. Comparing this expression with (A.74), the increase in social welfare upon imposing the optimal tax is given by $\frac{\pi_a\alpha_a}{2(\pi_s\alpha_s)^2}+O\left(\frac{1}{(\pi_s\alpha_s)^3}\right)$.

Next, the size of the unpatched population for this market structure is given by $u(\sigma^*|\tau)=v_a(\tau)-v_b(\tau)$, and the size of the automated patching population is given by $a(\sigma^*|\tau)=v_p(\tau)-v_a(\tau)$. Taking the limit of these as $\tau\to 0$, the sizes of the unpatched and automated populations under the status quo setting are given as $u_B=\frac{1}{\pi_s\alpha_s}+O\left(\frac{1}{(\pi_s\alpha_s)^3}\right)$ and $a_B=\frac{c_p-c_a}{\pi_a\alpha_a}-\frac{c_a}{1-\pi_a\alpha_a}+\frac{1}{\pi_s\alpha_s}+\frac{1-\pi_a\alpha_a(7-4\pi_a\alpha_a)}{2c_a(\pi_s\alpha_s)^2}+O\left(\frac{1}{(\pi_s\alpha_s)^3}\right)$. Comparing these to their respective population sizes under the optimal tax, $\tau=\tau_B$ (given in (A.73)), we establish that the size of the unpatched population decreases by $\frac{\pi_a\alpha_a(1-\pi_a\alpha_a)}{c_a(\pi_s\alpha_s)^2}+O\left(\frac{1}{(\pi_s\alpha_s)^3}\right)$ and the size of the automated patching population increases by $\frac{1}{\pi_s\alpha_s}+\frac{1-\pi_a\alpha_a(7-4\pi_a\alpha_a)}{2c_a(\pi_s\alpha_s)^2}+O\left(\frac{1}{(\pi_s\alpha_s)^3}\right)$.

Under both the status quo setting and under taxed patching rights, the total security loss from unpatched usage as a function of τ is given by $SL_B(\tau)=\int_{v_b(\tau)}^{v_a(\tau)}(v_a(\tau)-v_b(\tau))\pi_s\alpha_s v dv$ and total patching cost from automated patching is given by $AL_B(\tau)=\int_{v_a(\tau)}^{v_p(\tau)}c_a+\pi_a\alpha_a v dv$. Then the decrease in unpatched losses is given by $SL_B(0)-SL_B(\tau_B)=\frac{2\pi_a\alpha_a}{(\pi_s\alpha_s)^2}+O\left(\frac{1}{(\pi_s\alpha_s)^3}\right)$, and the increase in automated patching losses is given by $AL_B(\tau_B)-AL_B(0)=\frac{c_p}{\pi_s\alpha_s}+O\left(\frac{1}{(\pi_s\alpha_s)^2}\right)$. ■

Proof of Proposition 7: By Lemma 5, it can be seen that the magnitude of the optimal taxes imposed on patching rights significantly increases as c_a becomes less than or equal to $c_p-\pi_a\alpha_a$. The structure of the remaining proof is similar to the proof of Proposition 6.

(i) By Lemma 4 and Lemma 5, for sufficiently high $\pi_s\alpha_s$, we have the market structure $0<v_b<v_a<1$ being induced both under status quo setting without tax as well as under the optimal tax when $\frac{(\pi_a\alpha_a)^2}{1-\pi_a\alpha_a}<c_a\leq c_p-\pi_a\alpha_a$. The social welfare function of this case is given by (A.82). The welfare under the status quo case of no tax is given as

$$W_{ba,\text{ Status Quo}}=\frac{(1-c_a-\pi_a\alpha_a)^2}{2(1-\pi_a\alpha_a)}+O\left(\frac{1}{(\pi_s\alpha_s)^3}\right),\tag{A.85}$$

which is the limit of (A.82) as $\tau\to 0$. Comparing this expression with (A.84), the increase in social welfare upon imposing the optimal tax is given by $\frac{c_a}{4(1-\pi_a\alpha_a)\pi_s\alpha_s}+O\left(\frac{1}{(\pi_s\alpha_s)^2}\right)$.

Next, the size of the unpatched population for this market structure is given by $u(\sigma^*|\tau) = v_a(\tau) - v_b(\tau)$, and the size of the automated patching population is given by $a(\sigma^*|\tau) = 1 - v_a(\tau)$. Taking the limit of these as $\tau \to 0$, the sizes of the unpatched and automated populations under the status quo setting are given as $u_{ba} = \frac{1}{\pi_s\alpha_s} + O\left(\frac{1}{(\pi_s\alpha_s)^3}\right)$ and $a_{ba} = 1 - \frac{c_a}{1-\pi_a\alpha_a} + O\left(\frac{1}{(\pi_s\alpha_s)^3}\right)$. Comparing these to their respective population sizes under the optimal tax, $\tau = \tau_{ba}$ (given in (A.83)), we establish that the size of the unpatched population decreases by $\frac{1}{2\pi_s\alpha_s} + O\left(\frac{1}{(\pi_s\alpha_s)^2}\right)$ and the size of the automated patching population decreases by $\frac{1}{4(1-\pi_a\alpha_a)\pi_s\alpha_s} + O\left(\frac{1}{(\pi_s\alpha_s)^3}\right)$.

Under both the status quo setting and under taxed patching rights, the total security loss from unpatched usage as a function of τ is given by $SL_{ba}(\tau) = \int_{v_b(\tau)}^{v_a(\tau)} (v_a(\tau) - v_b(\tau))\pi_s\alpha_s v dv$ and total patching cost from automated patching is given by $AL_{ba}(\tau) = \int_{v_a(\tau)}^{1} c_a + \pi_a\alpha_a v dv$. Then the decrease in unpatched losses is given by $SL_{ba}(0) - SL_{ba}(\tau_{ba}) = \frac{3c_a}{4(1-\pi_a\alpha_a)\pi_s\alpha_s} + O\left(\frac{1}{(\pi_s\alpha_s)^2}\right)$, and the increase in automated patching losses is given by $AL_{ba}(0) - AL_{ba}(\tau_{ba}) = \frac{c_a}{4(1-\pi_a\alpha_a)^2\pi_s\alpha_s} + O\left(\frac{1}{(\pi_s\alpha_s)^2}\right)$.

(ii) Similarly, by Lemma 4 and Lemma 5, for sufficiently high $\pi_s\alpha_s$ and $c_a \le \min\left(c_p - \pi_a\alpha_a, \frac{(\pi_a\alpha_a)^2}{1-\pi_a\alpha_a}\right)$, we have the market structure $0 < v_b < v_a < 1$ being induced under status quo setting while $0 < v_a < v_b < 1$ is induced under the optimal tax. The welfare under the status quo case of no tax was found in the previous case. Comparing (A.85) with the welfare under the optimal tax (A.81), we find that the increase in social welfare upon imposing the optimal tax is given by $\frac{(c_a+\pi_a\alpha_a)^2}{4\pi_s\alpha_s} + O\left(\frac{1}{(\pi_s\alpha_s)^2}\right)$.

Next, the size of the unpatched population for this market structure under the optimal tax is given by $u(\sigma^*|\tau) = 1 - v_b(\tau)$, and the size of the automated patching population is given by $a(\sigma^*|\tau) = v_b(\tau) - v_a(\tau)$. Evaluating these under the optimal tax, $\tau = \tau_{ab}$ (given in (A.80)), the equilibrium sizes of the unpatched and automated patching populations under the optimal tax are $u_{ab} = \frac{c_a+\pi_a\alpha_a}{2\pi_s\alpha_s} + O\left(\frac{1}{(\pi_s\alpha_s)^2}\right)$ and $a_{ab} = \left(1 - \frac{c_a}{1-\pi_a\alpha_a}\right) - \frac{c_a+\pi_a\alpha_a}{2\pi_s\alpha_s} + O\left(\frac{1}{(\pi_s\alpha_s)^2}\right)$. From the previous case, the sizes of the unpatched and automated populations under the status quo setting are given as $u_{ba} = \frac{1}{\pi_s\alpha_s} + O\left(\frac{1}{(\pi_s\alpha_s)^3}\right)$ and $a_{ba} = 1 - \frac{c_a}{1-\pi_a\alpha_a} + O\left(\frac{1}{(\pi_s\alpha_s)^3}\right)$. Comparing these, we establish that the size of the unpatched population decreases by $\frac{2-c_a-\pi_a\alpha_a}{2\pi_s\alpha_s} + O\left(\frac{1}{(\pi_s\alpha_s)^2}\right)$ and the size of the automated patching population decreases by $\frac{c_a+\pi_a\alpha_a}{2\pi_s\alpha_s} + O\left(\frac{1}{(\pi_s\alpha_s)^2}\right)$ under the optimal tax.

Under taxed patching rights, the total security loss from unpatched usage as a function of τ is given by $SL_{ab}(\tau) = \int_{v_b(\tau)}^{1} (1 - v_b(\tau))\pi_s\alpha_s v dv$ and total patching cost from automated patching is given by $AL_{ab}(\tau) = \int_{v_a(\tau)}^{v_b(\tau)} c_a + \pi_a\alpha_a v dv$. Then the change in unpatched losses is given by $SL_{ba}(0) - SL_{ab}(\tau_{ab}) = \frac{1}{\pi_s\alpha_s}\left(\frac{c_a}{1-\pi_a\alpha_a} - \frac{1}{4}(c_a + \pi_a\alpha_a)^2\right) + O\left(\frac{1}{(\pi_s\alpha_s)^2}\right)$, and the increase in automated patching losses is given by $AL_{ba}(0) - AL_{ab}(\tau_{ab}) = \frac{(c_a+\pi_a\alpha_a)^2}{2\pi_s\alpha_s} + O\left(\frac{1}{(\pi_s\alpha_s)^2}\right)$. Note that $SL_{ba}(0) - SL_{ab}(\tau_{ab}) > 0$ if $c_a > \frac{2(1-\sqrt{1-\pi_a\alpha_a(1-\pi_a\alpha_a)}) - \pi_a\alpha_a(1-\pi_a\alpha_a)}{1-\pi_a\alpha_a}$ and $SL_{ba}(0) - SL_{ab}(\tau_{ab}) \le 0$ otherwise. ■